Topological implications in quantum tomography

Michael Wolf
NBI Copenhagen
TU Munich
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Topological implications in quantum tomography

David Reeb
NBI Copenhagen
TU Munich
Question: How many measurements/outcomes are necessary to identify a quantum state $\rho$ under prior information $\rho \in M$?

Setup:
• assume: prior info restricts to manifold $M$ of dimensionality $d_M$
• measure (i) $m$ expectation values or (ii) POVM with $m + 1$ outcomes:
  \[ h : M \to \mathbb{R}^m, \quad h(\rho)_i = \text{tr}[\rho A_i] \]

Goal:
• find minimal $m$ s.t. $h$ is injective (info complete for $M$)

Example: $M = \text{pure states in} \ C^d : \quad d_M = 2d - 2 \leq m \leq d^2 - 1$

[Flammia et al.]: $2d - 1 \leq m$
[Gross et al.]: efficient probabilistic scheme with $m = O(d(\log d)^2)$
Topological obstructions

Proposition: $h : M \rightarrow \mathbb{R}^m$, $h(\rho)_i = \text{tr}[\rho^{\otimes n} A_i]$

is info-complete for $M$ iff it is a topological embedding

Recipe for lower bounds on $m$:

show that topological properties of $M$ have no realization in too small dimensions $m$

Powerful toolboxes: homotopy, cohomology, etc.
Example 1: $M = \text{pure qubit states}$
$d_M = 2$

Observation: map from Bloch-SPHERE to $\mathbb{R}^2$ either discontinuous or not injective
i.e. $m > d_M$.

Corollary: $h : M \to \mathbb{R}^m$, $h(\rho)_i = \text{tr}[\rho^\otimes n A_i]$

is info-complete for $M$ iff it is so for all qubit states.

Borsuk-Ulam: If $m = 2$ then there exist two orthogonal states
which cannot be distinguished.
Example 2: \( M = \) pure states in \( \mathbb{C}^3 \) with \textbf{real} amplitudes
\[ |\psi\rangle = (x, y, z) \in \mathbb{R}^3 \]
\[ d_M = 2 \]

Observation: \( M \simeq \) real projective plane \( \mathbb{RP}^2 \)

Corollary: \( h : M \to \mathbb{R}^m, \quad h(\rho)_i = \text{tr}[\rho \otimes A_i] \)
is info-complete for \( M \) only if \( m \geq 4 \). \( m = 4 \) can be realized for \( n = 1 \).

proof idea: • non-orientability of \( \mathbb{RP}^2 \) implies self-intersections in \( \mathbb{R}^3 \)
• \((x, y, z) \mapsto (yz, xz, xy, x^2 - y^2)\) leads to \textbf{topological embedding}
Obstructions from differential topology

Proposition: With some assumptions on $M$, $h : M \to \mathbb{R}^m$, $h(\rho)_i = \text{tr}[\rho A_i]$ is info-complete for $M$ iff it is an embedding in the category of differential topology.

Assumptions: $M$ is smooth submanifold
- Union of tangent spaces is contained in \textit{`difference space`} $\{X | X = \lambda(M_1 - M_2), M_i \in M, \lambda > 0\}$

Lemma: This holds for $M = \mathbb{CP}^{d-1}, G(r, d-r)$

Powerful toolboxes for lower bounds on $m$:
- Atiyah Hirzebruch index theorem
- Chern’s results on dual Stiefel-Whitney classes
Pure states in $\mathbb{C}^d$

Proposition: The min $m$ for which $h : M \to \mathbb{R}^m$, $h(\rho)_i = \text{tr}[\rho A_i]$ can be info-complete satisfies

$$2d_M - 2\alpha < m \leq 2d_M - \alpha$$

where $\alpha = \text{number of 1's in binary expansion of } d - 1$

note: $\alpha \leq \log_2 d$, $d_M = 2d - 2$
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Remarks: • Analogous result for states with rank constraint (via Grassmannians)
In particular $m \leq 2d_M - 1$, $d_M = 2r(d - r)$
• Upper bounds via explicitly constructed observables
General upper bound

Let $M$ be a set with Minkowski dimension

$$D_M := \limsup_{\epsilon \to 0} \frac{\log N_\epsilon}{\log(1/\epsilon)}, \quad N_\epsilon = \min \text{ number of covering } \epsilon \text{ balls}$$

(note: $D_M = d_M$ for smooth manifolds)

Proposition: Almost every $h : M \to \mathbb{R}^m$, $h(\rho)_i = \text{tr}[\rho A_i]$ is info-complete for $M$ if $m > 2D_M$
Conclusion

• Topological properties of prior information are relevant for min $m$.

• $m$ can exceed the number of parameters necessary for description by a factor of two but not more.

• Results beat e.g. compressed sensing. However, we optimized $m$ irrespective of classical post-processing, robustness and verifyability of assumptions.

joint work of: Luca Mazzarella
               Teiko Heinosaari
               Michael Wolf

presentation: David Reeb

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contact: Michael Wolf (wolf.qit@gmail.com)