Quantum strategic game theory

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Why we are here?

- Understanding the power of quantum
  - Computation: quantum algorithms/complexity
  - Communication: quantum info. theory
  - …
- This work: game theory

Game: Two basic forms

strategic (normal) form

extensive form

Game: Two basic forms

strategic (normal) form

- n players: $P_1, \ldots, P_n$
- P_i has a set S_i of strategies
- P_i has a utility function $u_i: S \to \mathbb{R}$
  - $S = S_1 \times S_2 \times \cdots \times S_n$

Nash equilibrium

- Nash equilibrium: each player has adopted an optimal strategy, provided that others keep their strategies unchanged

Nash equilibrium

- Pure Nash equilibrium:
  a joint strategy $s = (s_1, \ldots, s_n)$ s.t. $\forall i$,
  $$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

- (Mixed) Nash equilibrium (NE):
  a product distribution $p = p_1 \times \ldots \times p_n$ s.t. $\forall i, s_i'$
  $$E_{s \leftarrow p}[u_i(s_i, s_{-i})] \geq E_{s \leftarrow p}[u_i(s_i', s_{-i})]$$

Correlated equilibrium

- Correlated equilibrium (CE): p s.t. $\forall i, s_i, s_i'$
  $$E_{s_{-i} \sim p(\cdot|s_i)}[u_i(s_i; s_{-i})] \geq E_{s_{-i} \sim p(\cdot|s_i)}[u_i(s_i'; s_{-i})]$$
- CE = NE ∩ {product distributions}

Nash and Aumann: two *Laureate of Nobel Prize in Economic Sciences*

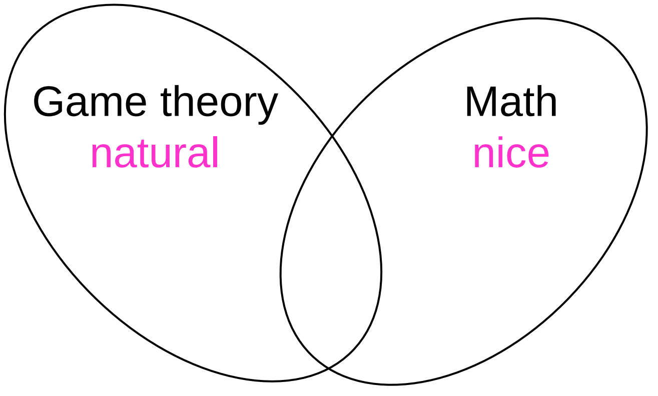
Why correlated equilibrium?

Traffic Light

	Cross	Stop
Cross	-100 -100	0 1
Stop	1 0	0 0

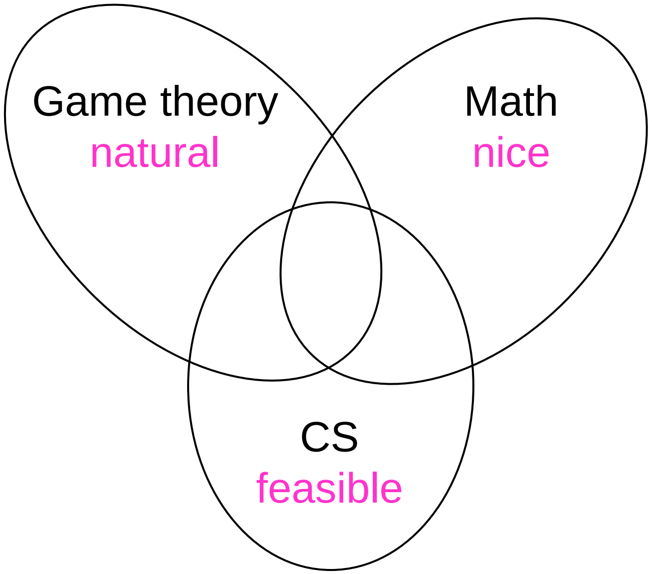
- 2 pure NE: one crosses and one stops. Payoff: (0,1) or (1,0)
  - Bad: unfair.
- 1 mixed NE: both cross w.p. 1/101.
  - Good: Fair
  - Bad: Low payoff: both ≃ 0.0001
  - Worse: Positive chance of crash
- CE: (Cross,Stop) w.p. ½, (Stop,Cross) w.p. ½
  - Fair, high payoff, 0 chance of crash.

Why correlated equilibrium?



- Set of correlated equilibria is convex.
- The NE are vertices of the CE polytope (in any non-degenerate 2-player game)
- All CE in graphical games can be represented by ones as product functions of each neighborhood.

Why correlated equilibrium?

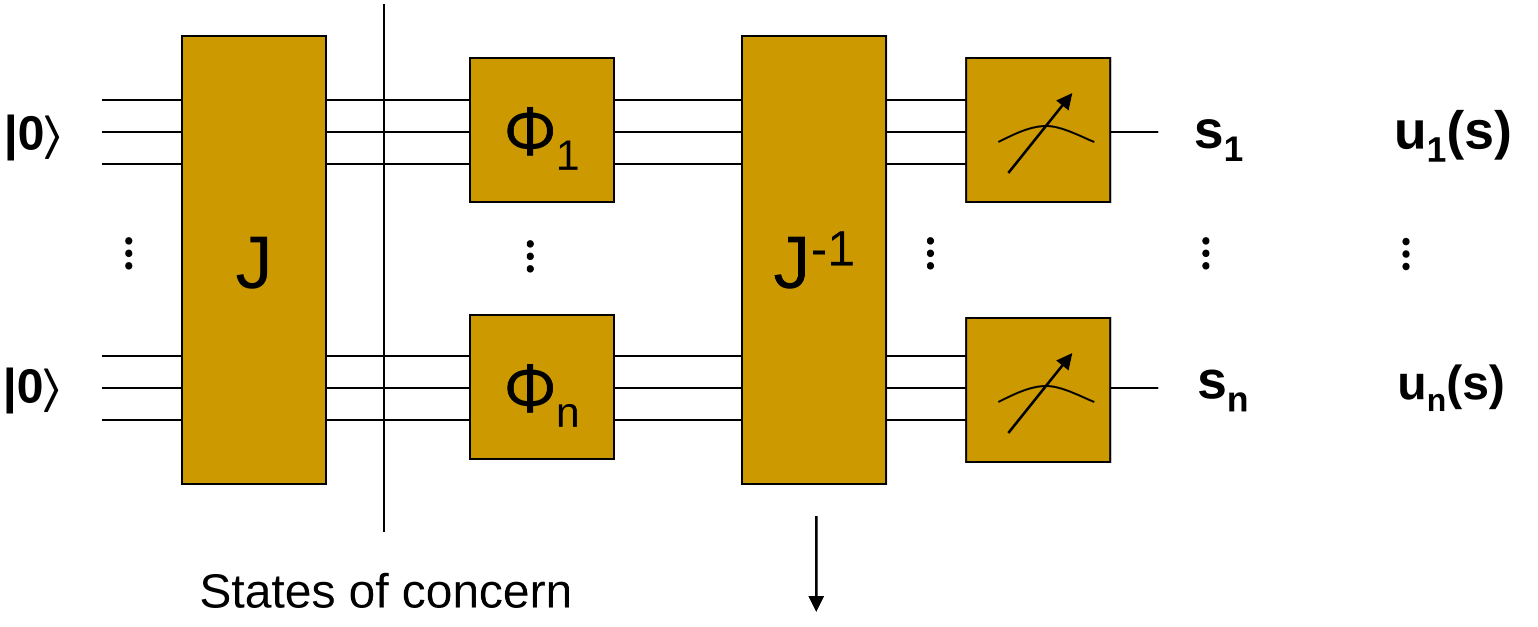


- [Obs] A CE can found in poly. time by LP.
- natural dynamics → approximate CE.
- A CE in graphical games can be found in poly. time.

“quantum games”

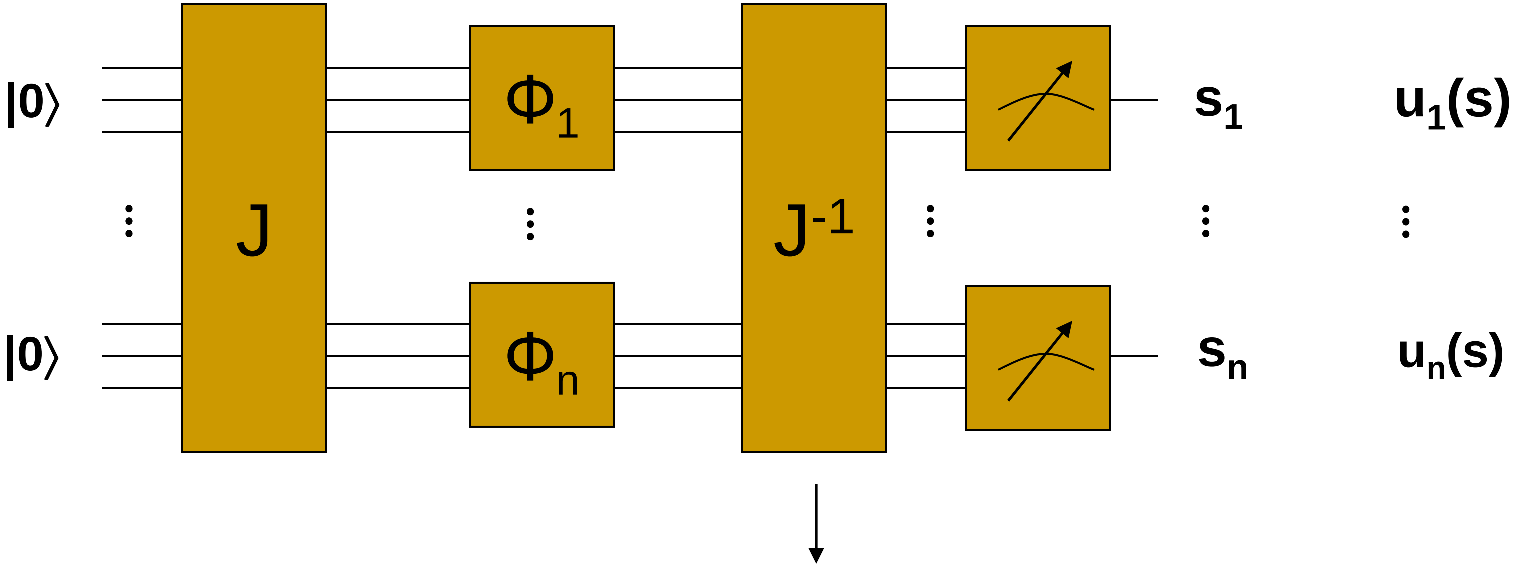
- Non-local games
- EWL-quantization of strategic games
  - J. Eisert, M. Wilkens, M. Lewenstein, *Phys. Rev. Lett.*, 1999.
- Others
  - Meyer’s Penny Matching
  - Gutoski-Watrous framework for refereed game

EWL model



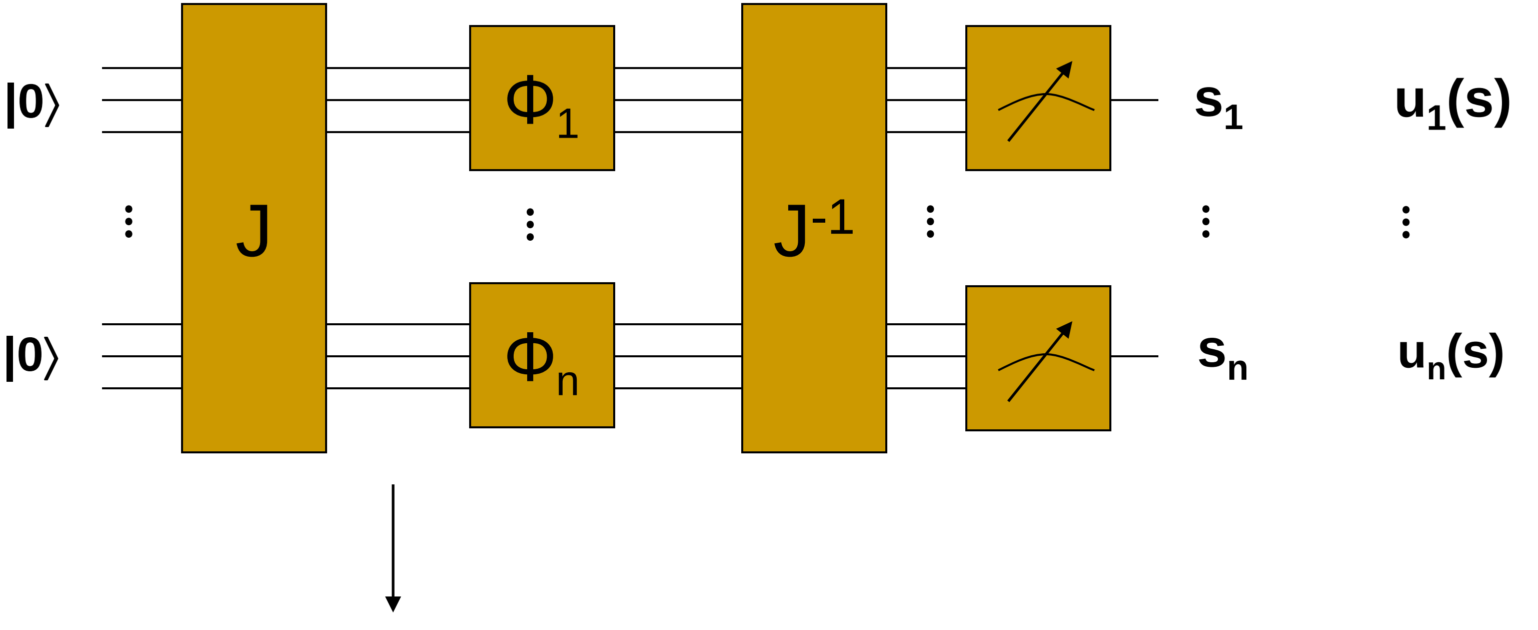
What's this classically?

EWL model



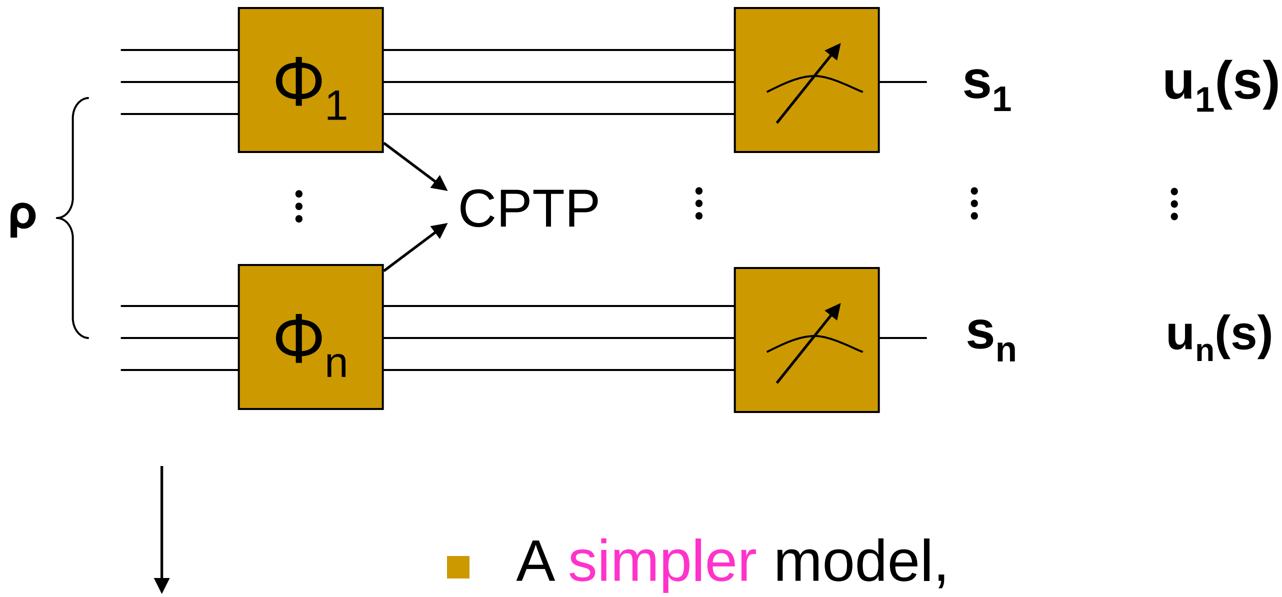
Classically we don't undo the sampling (or do any re-sampling) after players' actions.

EWL model



and consider the
state p at this point

Our model



and consider the state p at this point

- A simpler model,
- corresponding to classical games more precisely.

Other than the model

- Main differences than previous work in quantum strategic games:

- We consider general games of growing sizes.
  - Previous: specific games, usually 2*2 or 3*3
- We study quantitative questions.
  - Previous work: advantages exist?
  - Ours: How much can it be?

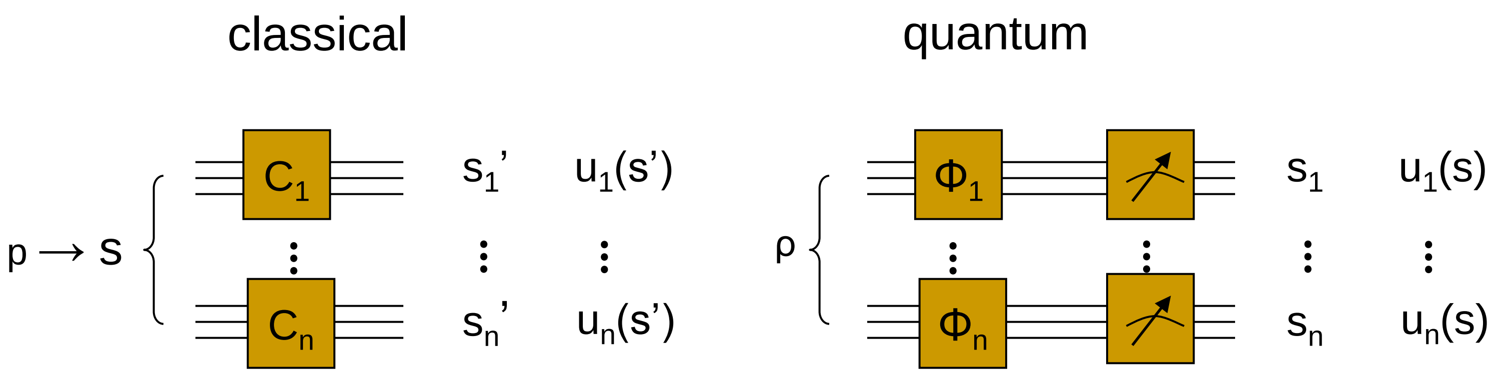
- *Central question: How much “advantage” can playing quantum provide?*

- Measure 1: Increase of payoff
- Measure 2: Hardness of generation

First measure: increase of payoff

- We will define natural correspondences between classical distributions and quantum states.
- And examine how well the equilibrium property is preserved.

Quantum equilibrium



classical equilibrium:

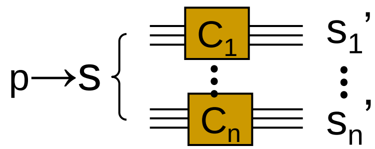
No player wants to do anything to the assigned strategy s_i, if others do nothing on their parts
- $p = p_1 \times \ldots \times p_n$: Nash equilibrium
- general p: correlated equilibrium

quantum equilibrium:

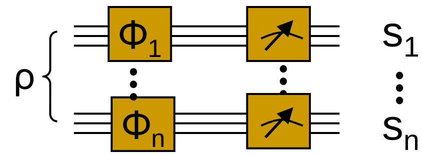
No player wants to do anything to the assigned strategy $\rho|_{H_i}$, if others do nothing on their parts
- $\rho = \rho_1 \times \ldots \times \rho_n$: quantum Nash equilibrium
- general ρ: quantum correlated equilibrium

Correspondence of classical and quantum states

classical



quantum



$p: p(s) = \rho_{ss}$ ← ρ

(measure in comp. basis)

p: distri. on S →

- $\rho_p = \sum_s p(s) \, |s\rangle\langle s|$ (classical mixture)
- $|\psi_p\rangle = \sum_s \sqrt{p(s)} \, |s\rangle$ (quantum superposition)
- $\forall \, \rho$ s.t. $p(s) = \rho_{ss}$ (general class)

Preservation of equilibrium?

Obs:
ρ is a quantum Nash/correlated equilibrium
$\Rightarrow$ p is a (classical) Nash/correlated equilibrium

classical | quantum

$p: p(s) = \rho_{ss}$ ← ρ

p → $\rho_p = \sum_s p(s)\, |s\rangle\langle s|$
$|\psi_p\rangle = \sum_s \sqrt{p(s)}\, |s\rangle$
$\forall\, \rho$ s.t. $p(s) = \rho_{ss}$

p	NE	CE
ρ_p		
$\lvert\psi_p\rangle$		
gen. ρ		

Question: Maximum *additive* and *multiplicative* increase of payoff (in a [0,1]-normalized game)?

Maximum additive increase

classical | quantum

$p: p(s) = \rho_{ss}$ ← ρ

p →
- $\rho_p = \sum_s p(s) |s\rangle\langle s|$
- $|\psi_p\rangle = \sum_s \sqrt{p(s)} |s\rangle$
- $\forall \rho$ s.t. $p(s) = \rho_{ss}$

Open: Improve on $|\psi_p\rangle$?

Additive

p	NE	CE
ρ_p	0	0
$\vert\psi_p\rangle$	0	1-Õ(1/log n)
gen. ρ	1-1/n (Optimal)	1-1/n (Optimal)

Question: Maximum *additive* and *multiplicative* increase of payoff (in a [0,1]-normalized game)?

Maximum multiplicative increase

classical		quantum
p: $p(s) = \rho_{ss}$	←	ρ
p	→	$\rho_p = \sum_s p(s)\, \lvert s\rangle\langle s\rvert$ $\lvert\psi_p\rangle = \sum_s \sqrt{p(s)}\, \lvert s\rangle$ $\forall\, \rho$ s.t. $p(s) = \rho_{ss}$

Open: Improve on $\lvert\psi_p\rangle$?

multiplicative

p	NE	CE
ρ_p	1	1
$\lvert\psi_p\rangle$	1	$\Omega(n^{0.585\ldots})$
gen. ρ	n	n

Optimal Optimal

Question: Maximum *additive* and *multiplicative* increase of payoff (in a [0,1]-normalized game)?

Optimization

- The maximum increase of payoff on $|\psi_p\rangle$ for a CE p:
  - $\sqrt{p_j}$ is short for the column vector $(\sqrt{p_{1j}}, \ldots, \sqrt{p_{nj}})^T$.

Primal: $\max \sum_{i,j\in[n]} a_{ij}\left(\sqrt{p_j}^T E_i \sqrt{p_j} - p_{ij}\right)$ (Var: $A, P, E_i \in \mathbb{R}^{n\times n}, i\in[n]$)

Non-concave

s.t. $0 \le a_{ij} \le 1, \quad \forall i,j\in[n]$ (The game is [0,1]-normalized.)

$\sum_{ij} p_{ij} = 1, \quad p_{ij} \ge 0, \quad \forall i,j\in[n]$ (p is a distribution.)

$\sum_j a_{ij}p_{ij} \ge \sum_j a_{i'j}p_{ij}, \quad \forall i,i',j\in[n]$ (p is a correlated equilibrium.)

$\sum_i E_i = I_n, \quad E_i \succeq 0, \quad \forall i\in[n]$ ($\{E_i\}$ is a POVM measurement.)

Dual(A,P): $\min \; Tr(Y) - \sum_{i,j\in[n]} a_{ij}p_{ij}$ (Var: $Y \in \mathbb{R}^{n\times n}$)

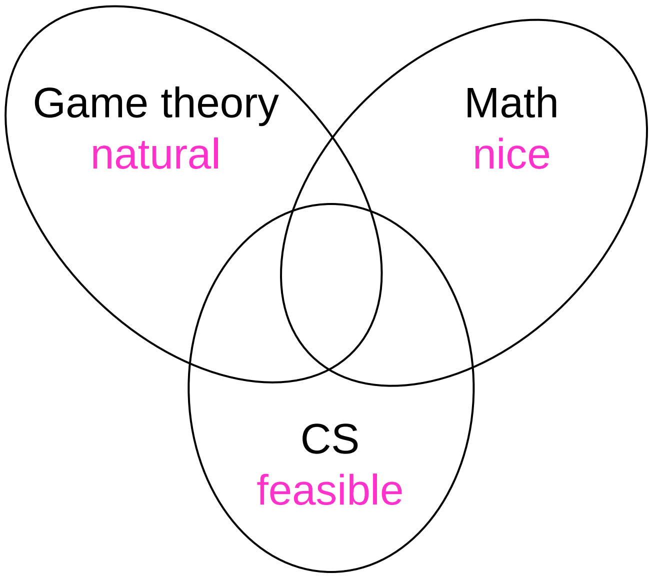
s.t. $Y \succeq \sum_{j\in[n]} a_{ij}\sqrt{p_j}\sqrt{p_j}^T, \quad \forall i\in[n]$

Small n and general case

- n=2:
  - Additive: $(1/\sqrt{2}) - 1/2 = 0.2071\ldots$
  - Multiplicative: 4/3.
- n=3:
  - Additive: $8/9 - 1/2 = 7/18 = 0.3888\ldots$
  - Multiplicative: 16/9.
- General n:
  - Tensor product
  - Carefully designed base case

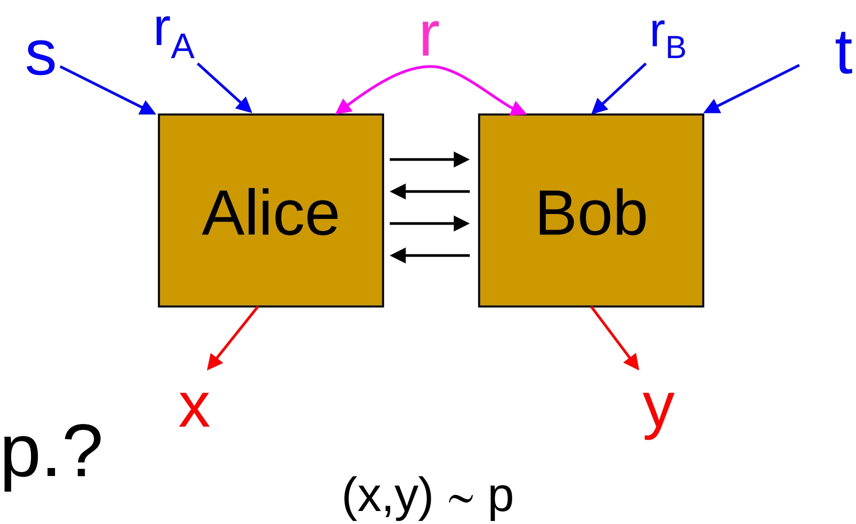
Second measure: hardness of generation

- Why care about generation?
- Recall the good properties of CE.
- But someone has to generate the correlation.
- Also very interesting on its own
  - Bell's inequality

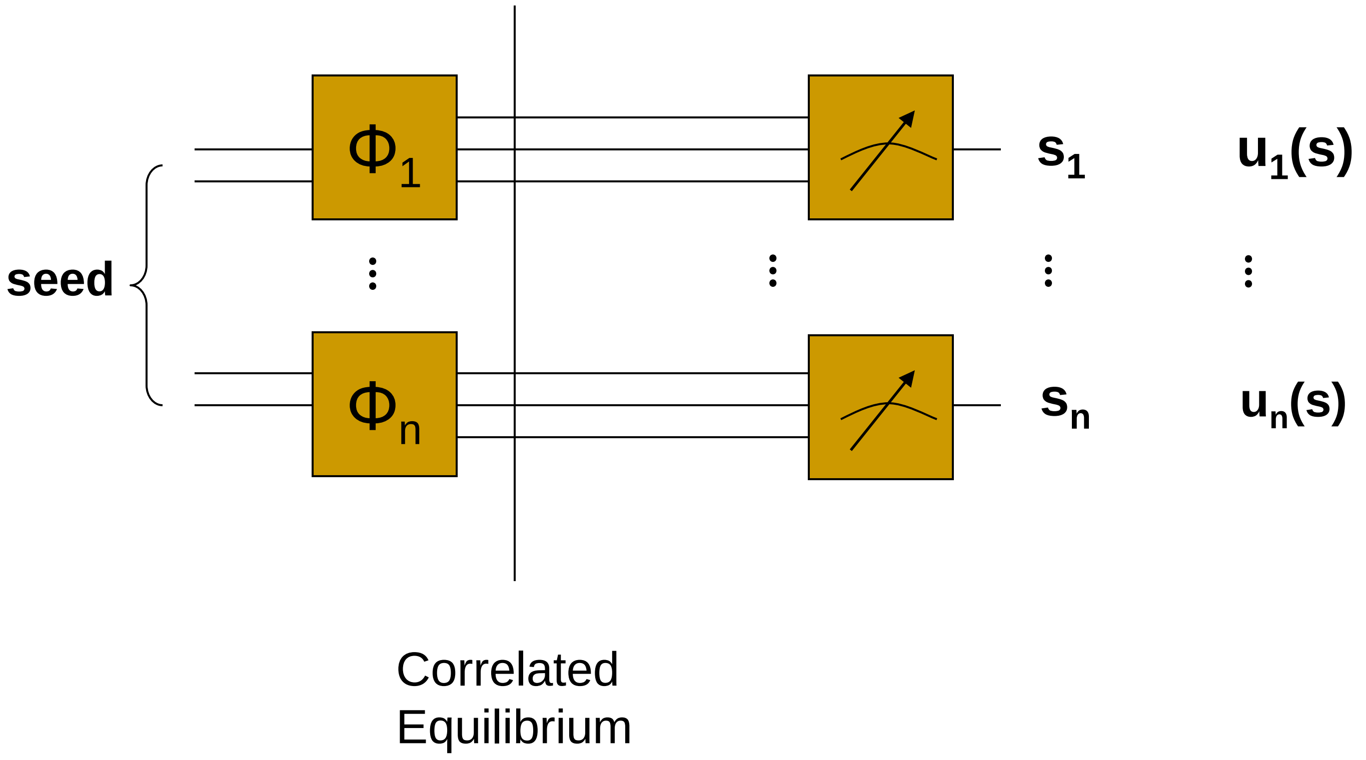


Correlation complexity

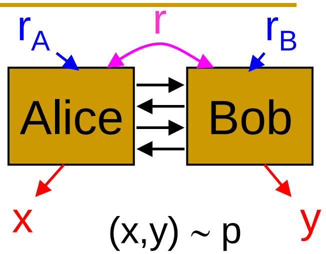
- Two players want to share a correlation.
- Need: shared resource or communication.
- Nonlocality? Comm. Comp.?
  - No private inputs here!
- Corr(p) = min shared resource needed
  - QCorr(p): entanglement   RCorr(p): public coins
- Comm(p) = min communication needed
  - QComm(p): qubits   RComm(p): bits



Correlation complexity back in games



Correlation complexity



- Question: Does quantum entanglement have advantage over classical randomness in generating correlation?
- [Obs] Comm(p) ≤ Corr(p) ≤ size(p)
  - Right inequality: share the target correlation.
- So unlike non-local games, one can always simulate the quantum correlation by classical.
- The question is the efficiency.

size(p) = length of string (x,y)

complexity-version of Bell's Theorem

Separation

[Conj] A random with

- [Thm] ∃ p=(X,Y) of size n, s.t.

  QCorr(p) = 1, RComm(p) ≥ log(n).

  n

Tools: rank and nonnegative rank

- [Thm] $\frac{1}{4}\log_2 \mathrm{rank}(P) \le \mathrm{QCorr}(p) \le \min_{Q:\, Q \circ \bar{Q} = P} \log_2 \mathrm{rank}(Q)$
  - $P = [p(x,y)]_{x,y}$
- [Thm] $\mathrm{RComm}(p) = \mathrm{RCorr}(p) = \lceil \log_2 \mathrm{rank}_+(P) \rceil$

- $\mathrm{rank}(M) = \min \{r: M = \sum_{k=1\ldots r} M_k,\ \mathrm{rank}(M_k)=1\}$
- *Nonnegative rank* (of a nonnegative matrix):
  $\mathrm{rank}_+(M) = \min \{r: M = \sum_{k=1\ldots r} M_k,\ \mathrm{rank}(M_k)=1,\ M_k \ge 0\}$
  - Extensively-studied in linear algebra and engineering. Many connections to (T)CS.

entrywise

Explicit instances

- Euclidean Distance Matrix (EDM):

$$Q(i,j) = c_i - c_j$$

  where $c_1, \ldots, c_N \in \mathbb{R}$.
  - rank(Q) = 2.
  - [Thm, BL09] $\mathrm{rank}_+(Q \circ Q) \geq \log_2 N$
  - [Conj, BL09] $\mathrm{rank}_+(Q \circ Q) = N$.
    (Even existing one Q implies 1 vs. n separation, the strongest possible)

Conclusion

- Model: natural, simple, rich
  - Non-convex programming; rank_+; comm. comp.

Thanks

- Next directions:
  - Improve the bounds (in both measures)
  - Efficient testing of QNE/QCE?
  - QCE $\leftarrow$ natural quantum dynamics?
  - Approximate Correlation complexity
    - [Shi-Z] $\exists$p: $QCorr_\varepsilon(p) = O(\log n)$, $RComm_\varepsilon(p) = \Omega(\sqrt{n})$
  - Characterize QCorr?
    - Mutual info? No! $\exists$p: $I_p = O(n^{-1/4})$, $QCorr(p) = \Theta(\log n)$

General n

- Construction: Tensor product.
- [Lem]

game	(u_1, u_2)	(u_1', u_2') →	$(u_1 \times u_1', u_2 \times u_2')$
CE	p	p' →	$p \times p'$
old u.	$u_1(\|\psi_p\rangle) = u$	$u_1(\|\psi_{p'}\rangle) = u'$ →	$u \cdot u'$
new utility	$u_1(\Phi\|\psi_p\rangle) = u_{new}$	$u_1(\Phi'\|\psi_{p'}\rangle) = u'_{new}$ →	$u_1((\Phi \otimes \Phi')(\|\psi_p\rangle \otimes \|\psi_{p'}\rangle)) = u_{new} u'_{new}$

Base case: additive increase

Worse than constant

- Using the result of n=2?
  - Additive: $\varepsilon_2^{\log 2(n)} - \varepsilon_1^{\log 2(n)} = 1/\text{poly}(n)$.
- Need: ε_2 and ε_1 very close to 1, yet still admitting a gap of ≈ 1 when taking power.
- New construction:

$$P = \begin{pmatrix} \sin^2(\epsilon) & \cos^2(\epsilon)\sin^2(\epsilon) \\ 0 & \cos^4(\epsilon) \end{pmatrix}$$

$$U_1 = \begin{pmatrix} \cos(\epsilon) & -\sin(\epsilon) \\ \sin(\epsilon) & \cos(\epsilon) \end{pmatrix}$$

$$\text{New } u = (1 - \sin^4(\epsilon))^{\log_2 n} = 1 - \Theta(1/\log n)$$

$$\text{Old } u = (1 - \sin^2(2\epsilon)/4)^{\log_2 n} = \Theta(1/\log n)$$