Near-Optimal and Explicit Bell Inequality Violations

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- Classical physics:
  - **Locality**: no faster than light influences.
  - **Realism**: values are determined before measurement.

- **EPR’35**: Quantum physics seems to violate local realism. Is it wrong or incomplete?

- **Bell’64**: Every local realistic theory must satisfy certain constraints (Bell Inequality).

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- Alice receives $x$ and Bob receives $y$, where $(x, y)$ are chosen from the distribution $\pi$.
- Alice outputs $a$ and Bob outputs $b$.
- A predicate specifies winning outputs.

**Goal**: maximize winning probability.

**Classical strategies**: functions $A(x), B(y)$.

- The classical value $\omega(G)$ is the maximum winning probability over all classical strategies.

**Quantum strategies**: shared entangled state; for each $x$ measurement $\{A^x_a\}$; for each $y$ $\{B^y_b\}$.

- Entangled value $\omega^*(G)$.
- $\omega^*_n(G)$ using entangled state of local dimension $\leq n$. 

- Space-like separated
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- **A Bell Inequality** is an upper bound on $\omega(G)$.
- **Violation**: $\omega^*(G)$ larger than $\omega(G)$.
  - Quantified by ratio $\frac{\omega^*(G)}{\omega(G)}$.
- **CHSH [Clauser, Horne, Shimony, Holt, 1969]**
  
  Classic example where $\frac{\omega_2^*(\text{CHSH})}{\omega(\text{CHSH})} \sim \frac{0.85}{0.75}$

- We want **large violations**!
  - Strong separation between quantum and classical worlds.
  - Typically easier to verify experimentally.

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Study violation as a function of:

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What is known?

How large can the ratio $\frac{\omega^*_n(G)}{\omega(G)}$ be?

**Upper Bounds:**
- [Junge, Palazuelos, Pérez-García, Villanueva, Wolf ’09]: with $n$-dimensional entanglement: $O(n)$.
- [Junge, Palazuelos ’10]: with $k$ possible outputs: $O(k)$.

**Lower Bounds:**
- [Folklore]: $n^\epsilon$ by parallel repetition of “magic square”.
- [Kempe, Regev, Toner ’08]: $n^{\epsilon'}$ from Unique Games.
- [JPPVW’09]: $\Omega(\sqrt{n}/(\log n)^2)$.
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Hidden Matching game

- Variant of “Hidden Matching” from communication complexity. [Bar-Yossef, Jayram, Kerenidis, STOC’04].
- $n$ outputs; entanglement dimension $n$.
- Violation of order $\sqrt{n}/\log n$.

Khot-Vishnoi game

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- Violation of order \( \sqrt{n} / \log n \).

**Khot-Vishnoi game**
- From an example of integrality gap for Unique Games [Khot, Vishnoi, FOCS’05] and a Quantum Rounding technique [Kempe, Regev, Toner, FOCS’08]
- \( n \) outputs; entanglement dimension \( n \).
- Violation of order \( n / (\log n)^2 \).
What are the inputs?

\[ x \]

1

0

1

1
What are the inputs?

Perfect Matching

\[ x \]

\[ M \]

1 \(\rightarrow\) (1, 2)

0 \(\rightarrow\) (1, 2)

1 \(\rightarrow\) (3, 4)

1
Hidden Matching communication game

They win if $v = x_i \oplus x_j$.

Thm: Classical winning probability is at most $\frac{1}{2} + O\left(c \sqrt{n}\right)$.

$[BJK'04]$ proved this for $c = \sqrt{n}$. 

[Image 2054x2092 to 2309x2219]
**Hidden Matching communication game**

The Hidden Matching game

They win if \( v = x_i \oplus x_j \).

**Thm:** Classical winning probability is at most \( \frac{1}{2} + O(c \sqrt{n}) \) (\([BJK'04]\) proved this for \( c = \sqrt{n} \)).

\[
x \in \{0, 1\}^n
\]

Perfect Matching

\[
M = (1, 2) \quad (3, 4)
\]
Hidden Matching communication game

\[ x \in \{0, 1\}^n \]

Perfect Matching \( M \)

\( (1, 2) \)

\( (3, 4) \)

\( c \text{ bits} \)
Hidden Matching communication game

1
0
1
1

\[ x \in \{0, 1\}^n \]

\[ v \in \{0, 1\}, \ (i, j) \in M \]

\[ \text{Perfect Matching} \]

\[ (1, 2) \]

\[ (3, 4) \]

\( c \text{ bits} \)
Hidden Matching communication game

They win if \( v = x_i \oplus x_j \).

\[ x \in \{0, 1\}^n \]

\[ M \]

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Perfect Matching

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They win if $v = x_i \oplus x_j$.

**Thm:** Classical winning probability is at most $\frac{1}{2} + O\left(\frac{c}{\sqrt{n}}\right)$ ([BJK’04] proved this for $c = \sqrt{n}$).
Hidden Matching non-local game

\[ x \in \{0,1\}^n \]

\[ a \in \{0,1\}^{\log n} \]

\[ d \in \{0,1\}, (i,j) \in M \]

Winning probability \(1\) with \(n\)-dimensional entanglement.

Classical bound \(1/2 + O(\log n / \sqrt{n})\).

Violation: \(\Omega(\sqrt{n \log n})\).
Hidden Matching *non-local* game

They win if \((a \cdot (i \oplus j)) \oplus d = x_i \oplus x_j\).
Hidden Matching non-local game

\[x \in \{0, 1\}^n\]

\[a \in \{0, 1\}^{\log n}\]

\[d \in \{0, 1\}, (i, j) \in M\]

They win if \((a \cdot (i \oplus j)) \oplus d = x_i \oplus x_j\).

Winning probability 1 with \(n\)-dimensional entanglement.


**Hidden Matching non-local game**

They win if \((a \cdot (i \oplus j)) \oplus d = x_i \oplus x_j\).

Winning probability 1 with \(n\)-dimensional entanglement.

Classical bound \(\frac{1}{2} + O\left(\frac{\log n}{\sqrt{n}}\right)\).
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Winning probability 1 with \(n\)-dimensional entanglement. Classical bound \(\frac{1}{2} + O\left(\frac{\log n}{\sqrt{n}}\right)\). **Violation:** \(\Omega\left(\frac{\sqrt{n}}{\log n}\right)\).
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Khot-Vishnoi game

$G(\{0, 1\}^n, \oplus)$
Khot-Vishnoi game

$G(\{0, 1\}^n, \oplus)$

Subgroup of all $n$ Hadamard codewords
**Khot-Vishnoi game**

For each $i$, set $z_i = 1$ with probability $\eta \in [0, 1/2]$ (we will choose $\eta$ later close to $1/2$).

$$G(\{0, 1\}^n, \oplus)$$

$$u \in \{0, 1\}^n$$

$$z \in \eta \{0, 1\}^n$$
Khot-Vishnoi game

\[ G(\{0, 1\}^n, \oplus) \]

- \( u \in \{0, 1\}^n \)
- \( z \in \eta \{0, 1\}^n \)

Winning condition:

\[ a \oplus b = z \]
Khot-Vishnoi game

The Khot-Vishnoi game

Winning condition: \( a \oplus b = z \)

\[ G(\{0, 1\}^n, \oplus) \]

\[ u \in \{0, 1\}^n \]

\[ z \in \eta \{0, 1\}^n \]

\[ x, y, H, u \oplus H, u \oplus z \oplus H \]
Khot-Vishnoi game

\[ G(\{0, 1\}^n, \oplus) \]

\[ u \in \{0, 1\}^n \]
\[ z \in \{0, 1\}^n \]
Khot-Vishnoi game

Winning condition: \( a \oplus b = z \).
Khot-Vishnoi game

\[ G(\{0, 1\}^n, \oplus) \]

\[ u \in \{0, 1\}^n \]
\[ z \in \eta \{0, 1\}^n \]

Winning condition: \( a \oplus b = z \).
Khot-Vishnoi - Quantum strategy

For any $n$ and $\eta \in [0, 1/2]$, there exists a quantum strategy that wins with probability at least $(1 - 2\eta)^2$.

- For $a \in \{0, 1\}^n$, define $|v^a\rangle = ((-1)^{a_i} / \sqrt{n})_{i \in [n]}$.
  - For all $a, b$, $\langle v^a, v^b \rangle = 1 - 2d(a, b)/n$
  - The vectors $\{v^a \mid a \in x\}$ are an orthonormal basis of $\mathbb{R}^n$.

- Quantum strategy (for Alice, similar for Bob):
  - Shared maximally entangled state, local dimension $n$.
  - On input $x$, projective measurement $\{v^a \mid a \in x\}$.
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- Probability to obtain $a, b$ is $\frac{\langle v^a, v^b \rangle^2}{n}$.
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- For inputs $x, y$, winning probability is
  \[
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  \]
- The overall winning probability is
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Khot-Vishnoi - Quantum strategy (2)

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- The overall winning probability is
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Khot-Vishnoi - Quantum strategy (2)

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Khot-Vishnoi - Classical bound

Every classical strategy has winning probability \( \leq \frac{1}{n^{\eta/(1-\eta)}} \)

- Fix strategy. Functions \( A, B : \{0, 1\}^n \to \{0, 1\} \).
  - \( A(u) = 1 \iff \) Alice's output on coset \( u \oplus H \) is \( u \).
  - \( \mathbb{E}_u[A(u)] = 1/n \) (Alice chooses one element per coset).
  - Players win \( \iff \sum_{h \in H} A(u \oplus h)B(u \oplus z \oplus h) = 1 \).
- Winning probability is \( \mathbb{E}[\sum_{u,z} A(u \oplus h)B(u \oplus z \oplus h)] \)
  \[
  = \sum_{h \in H} \mathbb{E}[A(u \oplus h)B(u \oplus z \oplus h)] = n \mathbb{E}[A(u)B(u \oplus z)]
  \]
- We have that \( \mathbb{E}[A(u)B(u \oplus z)] \leq \frac{1}{n^{1/(1-\eta)}} \)
  (proof by hypercontractivity, next slide).
- Theorem follows by noting that \( n \cdot \frac{1}{n^{1/(1-\eta)}} = \frac{1}{n^{\eta/(1-\eta)}} \).
Khot-Vishnoi - Classical bound

Every classical strategy has winning probability $\leq \frac{1}{n^{\eta/(1-\eta)}}$

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  $$= \sum_{h \in H} \mathbb{E}_u[A(u \oplus h) B(u \oplus z \oplus h)] = n \mathbb{E}_{u,z}[A(u) B(u \oplus z)]$$

- We have that $\mathbb{E}_{u,z}[A(u) B(u \oplus z)] \leq \frac{1}{n^{1/(1-\eta)}}$
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- Theorem follows by noting that \( n \cdot \frac{1}{n^{1/(1-\eta)}} = \frac{1}{n^{\eta/(1-\eta)}} \).
Every classical strategy has winning probability $\leq 1/n^{\eta/(1-\eta)}$

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- Winning probability is $E\left[\sum_{u,z} A(u \oplus h)B(u \oplus z \oplus h)\right]$

  $= \sum_{h \in H} E[A(u \oplus h)B(u \oplus z \oplus h)] = n \cdot E[A(u)B(u \oplus z)]$

- We have that $E[A(u)B(u \oplus z)] \leq \frac{1}{n^{1/(1-\eta)}}$
  (proof by hypercontractivity, next slide).

- Theorem follows by noting that $n \cdot \frac{1}{n^{1/(1-\eta)}} = \frac{1}{n^{\eta/(1-\eta)}}$. 
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noise operator

\[ \|T_{\rho}F\|_2 \leq \|F\|_{1+\rho^2} \]

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Setting $\eta = \frac{1}{2} - \frac{1}{\log n}$

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Conclusions and Open Problems

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- Reduce the number of inputs.
- Consider games with more than two players.
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