Constructing Quantum Network Coding Schemes from Classical Nonlinear Protocols

Hirotada Kobayashi
National Institute of Informatics

François Le Gall
The University of Tokyo

Harumichi Nishimura
Osaka Prefecture University

Martin Roetteler
NEC America Labs

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Classical Network Coding: Butterfly Graph

- two sources $s_1$ and $s_2$
- two targets $t_1$ and $t_2$

Goal:
- send $x$ to $t_1$
- send $y$ to $t_2$

- each edge has capacity 1 bit
Classical Network Coding: Butterfly Graph

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**Network Coding**
[Ahlswede, Cai, Li, Yeung, 00]
Quantum Network Coding

- two sources $s_1$ and $s_2$
- two targets $t_1$ and $t_2$

Goal:
- send $|\varphi_1\rangle$ to $t_1$
- send $|\varphi_2\rangle$ to $t_2$

- each edge has capacity 1 qubit
Quantum Network Coding: Results

On the butterfly graph:

[Hayashi 2007]
[Hayashi, Iwama, Nishimura, Raymond, Yamashita 2007]
[Winter, Leung, Oppenheim 2006]
Quantum Network Coding: Results

On the butterfly graph:

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• **Perfect** network coding is impossible:
  for all quantum protocols, the fidelities at nodes $t_1$ and $t_2$ are $< 1$
Quantum Network Coding: Results

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• Perfect network coding is impossible:
  for all quantum protocols, the
  fidelities at nodes $t_1$ and $t_2$ are $< 1$
• Imperfect network coding is possible:
  there exists a quantum protocol whose
  fidelities at nodes $t_1$ and $t_2$ are $> 1/2$
Quantum Network Coding: Results

On the butterfly graph:

- **Perfect** network coding is impossible:
  for all quantum protocols, the fidelities at nodes $t_1$ and $t_2$ are < 1
- **Imperfect** network coding is possible:
  there exists a quantum protocol whose fidelities at nodes $t_1$ and $t_2$ are > 1/2

Other works:  
- [Shi, Soljanin 2006]  
- [Kinji, Murao, Soeda, Turner 2010]  

Monday’s poster session
Quantum Network Coding: Results

On the butterfly graph:

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Other works:  
[Shi, Soljanin 2006]
[Kinji, Murao, Soeda, Turner 2010] ← Monday’s poster session
Statement of our Results
Our Setting

- We allow free classical communication between any pair of adjacent nodes

\(|\varphi_1\rangle\) one qubit

\(|\varphi_2\rangle\) one qubit
Our Setting

- we allow free classical communication between any pair of adjacent nodes.

Preliminary result: quantum perfect network coding is possible on the butterfly graph.
Our Setting

- we allow free classical communication between any pair of adjacent nodes

preliminary result

quantum perfect network coding is possible on the butterfly graph

general result

this is true for any graph
Our Setting

- we allow **free classical communication** between any pair of adjacent nodes

**Preliminary result**

quantum **perfect** network coding is possible on the butterfly graph

**General result**

this is true for any graph

reasonable hypothesis: classical communication is much cheaper than quantum communication
The Classical $k$-pair Problem

given: • a directed (acyclic) graph
• $k$ source nodes $s_1, \ldots s_k$
• $k$ target nodes $t_1, \ldots t_k$

goal: one bit $x_i$ has to be sent from $s_i$ to $t_i$
Main Result

Suppose that a given instance of the classical $k$-pair problem has a solution.

Classical protocol
Main Result

Suppose that a given instance of the classical $k$-pair problem has a solution. Then the associated quantum instance has a perfect solution if free classical communication is allowed.
Relation with our Previous Work

This result improves and generalizes our previous work

[KLNR 2009]

arXiv:0908.1457 and ICALP’09

<table>
<thead>
<tr>
<th>[KLNR’09]</th>
<th>This talk</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of bits of free classical communication sent per edge</td>
<td>polynomial</td>
</tr>
<tr>
<td>condition on the classical protocol</td>
<td>linear protocol</td>
</tr>
</tbody>
</table>

Note: there exist solvable classical \(k\)-pair problems for which no linear protocol exists [Dougherty, Freiling and Zeger 2005] [Riis 2003]
Illustration on the Butterfly Graph
Quantum Protocol

classical coding scheme

quantum simulation

Three steps:

I. node-by-node simulation

II. removal of internal registers

III. removal of initial registers

\[ y = x \oplus (x \oplus y) \]

\[ x = (x \oplus y) \oplus y \]
I. node-by-node simulation

classical copy node:

\[
\begin{align*}
&\hspace{1cm} z_1 \\
&\hspace{1cm} z_2 \\
&\hspace{1cm} z_3
\end{align*}
\]

quantum simulation:

\[
\begin{align*}
&Q_1 \\
&Q_2 \\
&Q_3
\end{align*}
\]

new register initialized to \(|0\rangle\)

two new registers initialized to \(|0\rangle\)

classical parity node:

\[
\begin{align*}
&\hspace{1cm} z_1 \\
&\hspace{1cm} z_2 \\
&\hspace{1cm} z_3
\end{align*}
\]

quantum simulation:

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\begin{align*}
&Q_1 \\
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new register initialized to \(|0\rangle\)

\[
\begin{align*}
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&Q_2 \\
&Q_3
\end{align*}
\]
I. node-by-node simulation

classical copy node:

\[ z \]
\[ z \]
\[ z \]

quantum simulation:

\[ Q_1 \]
\[ Q_2 \]
\[ Q_3 \]

two new registers initialized to \(|0\rangle\)

\[ z = 0, 1 \]

\[ Q_1: \]
\[ Q_2: \]
\[ Q_3: \]

classical parity node:

\[ z_1 \oplus z_2 \]

quantum simulation:

\[ Q_1 \]
\[ Q_2 \]
\[ Q_3 \]

new register initialized to \(|0\rangle\)

\[ z_1, z_2 = 0, 1 \]

\[ Q_1: \]
\[ Q_2: \]
\[ Q_3: \]
I. node-by-node simulation: details

initial state: $|x\rangle_{S_1} |y\rangle_{S_2}$

$x, y \in \{0, 1\}$
I. node-by-node simulation: details

initial state: $|x\rangle_{S_1}|y\rangle_{S_2}$

$x, y \in \{0, 1\}$
I. node-by-node simulation: details

initial state: \( |x\rangle_{S_1} |y\rangle_{S_2} \)

\( x, y \in \{0, 1\} \)

\[
\begin{align*}
S_1: & \quad |x\rangle  \\
R_1: & \quad |0\rangle  \\
R_2: & \quad |0\rangle  \\
\end{align*}
\]
I. node-by-node simulation: details

initial state: $|x\rangle S_1 |y\rangle S_2$

basis state (for now)

$x, y \in \{0, 1\}$

S₁: $|x\rangle$
R₁: $|0\rangle$
R₂: $|0\rangle$

S₂: $|x\rangle$
R₂: $|x\rangle$
R₁: $|x\rangle$
I. node-by-node simulation: details

initial state: $|x\rangle s_1 |y\rangle s_2$

$x, y \in \{0, 1\}$

basis state (for now)

\[
\begin{align*}
S_1 : & |x\rangle \\
R_1 : & |0\rangle \\
R_2 : & |0\rangle \\
S_2 : & |y\rangle \\
R_3 : & |0\rangle \\
R_4 : & |0\rangle \\
\end{align*}
\]
I. node-by-node simulation: details

initial state: $|x\rangle_{S_1} |y\rangle_{S_2}$

$x, y \in \{0, 1\}$

I. node-by-node simulation: details

initial state: $|x\rangle_{S_1} |y\rangle_{S_2}$

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initial state: $|x\rangle_{S_1} |y\rangle_{S_2}$

$x, y \in \{0, 1\}$
I. node-by-node simulation: details

initial state: \( |x\rangle_{S_1} |y\rangle_{S_2} \)

basis state (for now)

\( x, y \in \{0, 1\} \)

\[
\begin{array}{c c c c}
S_1 : & |x\rangle & - & - & \ldots & - & - & |x\rangle \\
R_1 : & 0\rangle & - & - & \ldots & - & - & \ldots & |x\rangle \\
R_2 : & 0\rangle & - & - & \ldots & - & - & \ldots & \ldots & |x\rangle \\
S_2 : & |y\rangle & - & - & \ldots & - & - & \ldots & |y\rangle \\
R_3 : & 0\rangle & - & - & \ldots & - & - & \ldots & \ldots & |y\rangle \\
R_4 : & 0\rangle & - & - & \ldots & - & - & \ldots & \ldots & |y\rangle \\
R_5 : & 0\rangle & - & - & \ldots & - & - & \ldots & \ldots & |x \oplus y\rangle \\
\end{array}
\]
I. node-by-node simulation: details

initial state: \( |x\rangle_{S_1} |y\rangle_{S_2} \)

\( x, y \in \{0, 1\} \)

### Diagram

- **Initial State:** \( |x\rangle_{S_1} |y\rangle_{S_2} \)
- **Basis State:** (for now)

### Table

<table>
<thead>
<tr>
<th>Node</th>
<th>Initial State</th>
<th>Final State</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>( x )</td>
<td>( x )</td>
</tr>
<tr>
<td>( R_1 )</td>
<td>( 0 )</td>
<td>( x )</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>( 0 )</td>
<td>( x )</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>( y )</td>
<td>( y )</td>
</tr>
<tr>
<td>( R_3 )</td>
<td>( 0 )</td>
<td>( y )</td>
</tr>
<tr>
<td>( R_4 )</td>
<td>( 0 )</td>
<td>( y )</td>
</tr>
<tr>
<td>( R_5 )</td>
<td>( 0 )</td>
<td>( x \oplus y )</td>
</tr>
</tbody>
</table>
I. node-by-node simulation: details

initial state: $|x\rangle_{S_1} |y\rangle_{S_2}$

$x, y \in \{0, 1\}$

$S_1: |x\rangle$ $\rightarrow$ $|x\rangle$
$R_1: |0\rangle$ $\rightarrow$ $|x\rangle$
$R_2: |0\rangle$ $\rightarrow$ $|x\rangle$
$S_2: |y\rangle$ $\rightarrow$ $|y\rangle$
$R_3: |0\rangle$ $\rightarrow$ $|y\rangle$
$R_4: |0\rangle$ $\rightarrow$ $|y\rangle$
$R_5: |0\rangle$ $\rightarrow$ $|x \oplus y\rangle$
$R_6: |0\rangle$ $\rightarrow$ $|x \oplus y\rangle$
$R_7: |0\rangle$ $\rightarrow$ $|x \oplus y\rangle$
I. node-by-node simulation: details

initial state: $|x\rangle_{S_1} |y\rangle_{S_2}$

$x, y \in \{0, 1\}$
I. node-by-node simulation: details

initial state: \( |x\rangle S_1 |y\rangle S_2 \)

\( x, y \in \{0, 1\} \)
initial state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |y\rangle_{S_2} \]

final state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |x\rangle_{R_1} |x\rangle_{R_2} |y\rangle_{S_2} |y\rangle_{R_3} |y\rangle_{R_4} \otimes |x \oplus y\rangle_{R_5} |x \oplus y\rangle_{R_6} |x \oplus y\rangle_{R_7} |x\rangle_{T_1} |y\rangle_{T_2} \]
initial state: $\sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle |S_1\rangle |y\rangle |S_2\rangle$

final state: $\sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle |S_1\rangle |x\rangle |R_1\rangle |x\rangle |R_2\rangle |y\rangle |S_2\rangle |y\rangle |R_3\rangle |y\rangle |R_4\rangle \otimes |x \oplus y\rangle |R_5\rangle |x \oplus y\rangle |R_6\rangle |x \oplus y\rangle |R_7\rangle |x\rangle |T_1\rangle |y\rangle |T_2\rangle$

ideal state: $\sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle |T_1\rangle |y\rangle |T_2\rangle$
initial state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |y\rangle_{S_2} \]

final state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |x\rangle_{R_1} |x\rangle_{R_2} |y\rangle_{S_2} |y\rangle_{R_3} |y\rangle_{R_4} \otimes |x \oplus y\rangle_{R_5} |x \oplus y\rangle_{R_6} |x \oplus y\rangle_{R_7} |x\rangle_{T_1} |y\rangle_{T_2} \]

ideal state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{T_1} |y\rangle_{T_2} \]

---

S1: |x⟩ R1: 0⟩ R2: 0⟩ S2: 0⟩ T1: 0⟩ R3: 0⟩ R4: 0⟩ R5: 0⟩ R6: 0⟩ R7: 0⟩ T2: 0⟩
II. removal of internal registers

A TRICK:

\[ H \xrightarrow{a} Z^a \]

- measurement in basis \(\{|0\rangle, |1\rangle\}\)
- outcome: \(a \in \{0, 1\}\)
II. removal of internal registers

A TRICK:

\[ \sum_{z=0,1} \beta_z |z\rangle |z\rangle \]

\[ \text{measurement in basis } \{|0\rangle, |1\rangle\} \]

outcome: \( a \in \{0, 1\} \)

\[ \sum_{z=0,1} \beta_z |z\rangle |a\rangle \]

disregarded
II. removal of internal registers

A TRICK:

\[ \sum_{z=0,1} \beta_z |z\rangle |0\rangle + (-1)^z |1\rangle \sqrt{2} \]

\[ \sum_{z=0,1} \beta_z |z\rangle |0\rangle \text{ if } a = 0 \]
\[ \sum_{z=0,1} \beta_z (-1)^z |z\rangle |1\rangle \text{ if } a = 1 \]

\[ \sum_{z=0,1} \beta_z |z\rangle |a\rangle \text{ disregarded} \]
II. removal of internal registers

ANOTHER TRICK:

\[ \sum_{x,y=0,1} \gamma_{xy} |x\rangle |y\rangle |0\rangle \text{ if } c = 0 \]
\[ \sum_{x,y=0,1} \gamma_{xy} (-1)^{x \oplus y} |x\rangle |y\rangle |1\rangle \text{ if } c = 1 \]

\[ |x\rangle \]
\[ |y\rangle \]
\[ |x \oplus y\rangle \]
\[ H \]
\[ \perp c \]
\[ Z^c \]

\[ \sum_{x,y=0,1} \gamma_{xy} |x\rangle |y\rangle |x \oplus y\rangle \]

\[ \sum_{x,y=0,1} \gamma_{xy} |x\rangle |y\rangle |c\rangle \]
II. removal of internal registers

idea: phases can always be corrected at the previous node

$S_1: \langle x \rangle$
$R_1: \langle x \rangle$
$R_2: \langle x \rangle$
$S_2: \langle y \rangle$
$R_3: \langle y \rangle$
$R_4: \langle y \rangle$
$R_5: \langle x \oplus y \rangle$
$R_6: \langle x \oplus y \rangle$
$R_7: \langle x \oplus y \rangle$
$T_1: \langle x \rangle$
$T_2: \langle y \rangle$
II. removal of internal registers

idea: phases can always be corrected at the
previous node

\[
\sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |x\rangle_{R_1} |x\rangle_{R_2} |y\rangle_{S_2} |y\rangle_{R_3} |y\rangle_{R_4} \otimes
\]

\[
|x \oplus y\rangle_{R_5} |\alpha\rangle_{R_6} |x \oplus y\rangle_{R_7} |x\rangle_{T_1} |y\rangle_{T_2}
\]

\[
\quad \quad \quad : 1 \text{ bit}
\]

**Nodes:**
- **S1:** \( |x\rangle \)
- **R1:** \( |x\rangle \)
- **R2:** \( |x\rangle \)
- **S2:** \( |y\rangle \)
- **R3:** \( |y\rangle \)
- **R4:** \( |y\rangle \)
- **R5:** \( |x \oplus y\rangle \)
- **R6:** \( |x \oplus y\rangle \)
- **R7:** \( |x \oplus y\rangle \)
- **T1:** \( |x\rangle \)
- **T2:** \( |y\rangle \)
II. removal of internal registers

idea: phases can always be corrected at the previous node

$$\sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |y\rangle_{S_2} |R_1\rangle |R_2\rangle |y\rangle_{R_3} |y\rangle_{R_4} \otimes |x \oplus y\rangle_{R_5} |\alpha\rangle_{R_6} |x \oplus y\rangle_{R_7} |x\rangle_{T_1} |y\rangle_{T_2}$$

$S_1: |x\rangle$
$R_1: |x\rangle$
$R_2: |x\rangle$
$S_2: |y\rangle$
$R_3: |y\rangle$
$R_4: |y\rangle$
$R_5: |x \oplus y\rangle$
$R_6: |x \oplus y\rangle$
$R_7: |x \oplus y\rangle$
$T_1: |x\rangle$
$T_2: |y\rangle$
II. removal of internal registers

idea: phases can always be corrected at the previous node

\[
\sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_S |y\rangle_S |x\rangle_R_1 |x\rangle_R_2 |y\rangle_S |y\rangle_R_3 |y\rangle_R_4 \otimes
\]

\[
|x \oplus y\rangle_{R_5} | \alpha \rangle_{R_6} |x \oplus y\rangle_{R_7} |x\rangle_T_1 |y\rangle_T_2
\]
II. removal of internal registers

idea: phases can always be corrected at the previous node

\[ R_5 \text{ was created using } R_2 \text{ and } R_3 \]
II. removal of internal registers

idea: phases can always be corrected at the previous node

R₅ was created using R₂ and R₃
II. removal of internal registers

Idea: phases can always be corrected at the previous node

R₅ was created using R₂ and R₃
III. removal of initial registers

Current state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} \left| x \right\rangle_{S_1} \left| y \right\rangle_{S_2} \left| x \right\rangle_{T_1} \left| y \right\rangle_{T_2} \]

Ideal state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} \left| x \right\rangle_{T_1} \left| y \right\rangle_{T_2} \]

\[ S_1 : \left| x \right\rangle \]
\[ S_2 : \left| y \right\rangle \]
\[ T_1 : \left| x \right\rangle \]
\[ T_2 : \left| y \right\rangle \]
III. removal of initial registers

exception: phases are corrected at the target nodes

current state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{S_1} |y\rangle_{S_2} |x\rangle_{T_1} |y\rangle_{T_2} \]

ideal state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x\rangle_{T_1} |y\rangle_{T_2} \]
III. removal of initial registers

exception: phases are corrected at the target nodes

current state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x \rangle_1 |y \rangle_2 |x \rangle_3 |y \rangle_4 |x \rangle_5 |y \rangle_6 \]

ideal state: \[ \sum_{x,y \in \{0,1\}} \alpha_{xy} |x \rangle_1 |y \rangle_2 \]

\[ S_1 : |x \rangle \quad H \quad \alpha \quad k \]
\[ S_2 : |y \rangle \quad H \quad \alpha \quad h \]
\[ T_1 : |x \rangle \quad Z^k \]
\[ T_2 : |y \rangle \quad Z^h \]
III. removal of initial registers

exception: phases are corrected at the target nodes

equivalently:

![Diagram showing the process of removing initial registers with the notation of qubits and bits being sent along edges.](attachment:image.png)

one qubit + two bits sent per edge
Main Theorem

Suppose that a given instance of the classical $k$-pair problem has a solution. Then the associated quantum instance has a perfect solution if free classical communication is allowed.

one qubit + two bits sent per edge
General Quantum Protocol

classical coding scheme

quantum simulation

Three steps:

I. node-by-node simulation

II. removal of internal registers

III. removal of initial registers
Conclusions

- Without additional resources, perfect quantum network coding is impossible in general.

- With free classical communication, perfect quantum coding is possible whenever classical coding is feasible:
  - this works even for nonlinear classical schemes
  - at most two bits of classical communication are sent per edge

- Our proof is constructive: efficient construction of a quantum perfect transmission protocol from any classical coding scheme.