Information propagation for interacting particle systems

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joint work with Sarah Harrison, Tobias Osborne, and Jens Eisert
Introduction

- How fast can **information propagate** in physical systems?
  - Obvious answer: **Relativity ➞ No faster than the speed of light!**

- This is true, but ...
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  ... e.g. in classical mechanical systems, information propagates **at a speed of sound**, without the need for relativistic arguments!

- This speed can be understood from the microscopic model, using
  - that it is **local**
  - that the interactions have **bounded strength**
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  - that it is local
  - that the interactions have bounded strength

⇒ Finite propagation speed can be understood non-relativistically!
Quantum mechanical systems

- What about quantum mechanical systems?
- **Quantum spin systems:**

\[ H = \sum_{\langle j, k \rangle} h_{jk} ; \quad \| h_{jk} \|_{op} \leq J \] : local Hamiltonian of bounded strength
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• Lieb-Robinson bounds:

[Lieb & Robinson '72, Hastings '04, Nachtergaele & Sims '06]

\[ \| [A(t), B] \| \leq c \| A \| \| B \| \exp[-(L - vt)/\xi] \]

Lieb-Robinson velocity \( v = c_G J \) depends on graph
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Lieb-Robinson velocity \( v = c_G J \)

- question of fundamental interest
- propagation speed of perturbations
- facilitates simulation of dynamics
- imaginary time \( \Rightarrow \) exponential decay of correlations

[Lieb & Robinson '72, Hastings '04, Nachtergaele & Sims '06]
Bosonic systems

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  (→ in particular, chains of quantum oscillators)
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- What about **systems of interacting particles**, such as bosons?
  
  (→ in particular, chains of *quantum* oscillators)

- canonical example: **Bose-Hubbard model**:

  \[
  H_{BH} = -\tau \sum_{<j,k>} (\hat{a}_j^\dagger \hat{a}_k^\dagger + \hat{a}_k \hat{a}_j) + U \sum_j \hat{n}_j (\hat{n}_j - 1)
  \]

  \[
  \hat{a}_j : \text{annihilate a particle at site } j \\
  \hat{a}_j^\dagger : \text{create a particle at site } j \\
  \hat{a}_j |n\rangle = \sqrt{n} |n - 1\rangle \quad \leftrightarrow \quad \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j : \text{counts particles at site}
  \]
What is the problem with bosons?

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- Lieb-Robinson bound does not apply:

\[ a_j^{\dagger} a_k | n_j - 1, n_k \rangle = \sqrt{n_j n_k} | n_j, n_k - 1 \rangle \]

⇒ hopping term \( a_j^{\dagger} a_k \) unbounded (or only by \( \| a_j^{\dagger} a_k \| \leq N_{\text{tot}} \))

⇒ Lieb-Robinson velocity \( v \propto \| h_{jk} \| \sim N_{\text{tot}} \)
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- examples where \( n_k \) and thus \( \nu \) grow unboundedly exist! \[ \text{[Gross & Eisert 2009]} \]
  \( \Rightarrow \) need constraints on Hamiltonian (e.g. particle number conserving)
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- examples where \( n_k \) and thus \( v \) grow unboundedly exist! \[\text{[Gross & Eisert 2009]}\]

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- hopping rate (and thus \( v \)) will depend on the filling of the lattice:

  ⇒ need **constraints on initial state**

- Note: bounds exist for quadr. Hamiltonians and certain perturbations thereof \[\text{[Nachtergaele, Raz, Schlein, Sims 2009]}\]
Idea: Restrict to relevant models

• Aim: propagation speed for Bose-Hubbard type models

• How can we obtain a meaningful propagation speed?
  - restrict to certain **initial states of interest** *(which allow for finite speed of propagation)*
  - only keep track of **relevant information** *(how do particles propagate)*
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Propagation of particles in the Hubbard model

- Generalized Hubbard model:

\[ H_{BH} = - \sum_{j<k} \tau_{jk} (\hat{a}_{j}^{\dagger} \hat{a}_{k} + \hat{a}_{k}^{\dagger} \hat{a}_{j}) + f(\hat{n}_{1}, \ldots, \hat{n}_{L}) \]

- consider time evolution \( \dot{\rho}(t) = -i[H_{BH}, \rho(t)] \) from initial state \( \rho(0) \)
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• consider only the **expected number of bosons per site**:

\[ \alpha_j(t) := \text{tr}[\hat{n}_j \rho(t)] \]
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- \( \dot{\rho}(t) = -i[H, \rho(t)] \) ⇒ differential inequality \( \dot{\alpha}_j(t) \leq 2 \sum_{\langle j, k \rangle} \tau_{jk} [\alpha_j(t) \alpha_k(t)]^{1/2} \)

⇒ worst-case upper bound \( \gamma_j(t) \geq \alpha_j(t) \) evolves according to:

\[ \dot{\gamma}_j(t) = 2 \sum_{\langle j, k \rangle} \tau_{jk} (\gamma_j(t) + \gamma_k(t)) \quad \text{(linearized)} \]
Obtaining a speed limit

- bound $\gamma_j(t) \geq \alpha_j(t) \Rightarrow$ worst-case solution for propagation

$\bar{\alpha}(t) \leq e^{Mt} \bar{\alpha}(0)$ with $M$ the “adjacency matrix”
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with $M$ the “adjacency matrix”

- $M$ banded matrix $\Rightarrow e^{Mt}$ decays exponentially away from diagonal

$$[e^{Mt}]_{jk} \leq c e^{\nu t - d(j,k)}$$

with $\nu = c_G \tau$

[Benzi & Golub '99]
Obtaining a speed limit

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- $M$ banded matrix $\Rightarrow e^{Mt}$ decays exponentially away from diagonal
  \[ \left[ e^{Mt} \right]_{jk} \leq c e^{vt-d(j,k)} \]
  with $v = c_G \tau$

- together:
  \[ \alpha_j(t) \leq CN_0 e^{vt-l} \]
  $v = c_G \tau$

$\Rightarrow$ speed independent of particle number!
Speed limit for interacting particles

\[ \alpha_j(t) \leq C N_0 e^{\nu t - l} \text{ where } \nu \propto \tau \]

• Proof idea: Study evolution of worst case bound on \( \alpha_j(t) \):
Speed limit for interacting particles

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![Graph showing the evolution of \( \alpha_j(t) \) over time, with a peak and a time-evolving exponential decay.](image)
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- argument works for **any Hubbard-type model** on **any graph**
Proof idea: Study evolution of worst case bound on $\alpha_j(t)$:

- Argument works for any Hubbard-type model on any graph
- Extension possible to
  - higher moments of particle number
  - arbitrary local operators
  - operators acting on larger blocks (up to log-size)
  … by iteratively bounding those quantities by $\alpha_j(t)$.
Extensions

• can be extended to **several species** of particles, **fermions**, **Bose-Fermi mixtures**, and even **anyons**:

\[ H = - \sum_{\langle j,k \rangle,s} \tau_{jk}(\hat{a}_{j,s}^{\dagger} \hat{a}_{k,s} + \hat{a}_{k,s}^{\dagger} \hat{a}_{j,s}) + f(\{n_{j,s}\}_{j,s}) \]

→ can be understood as **hopping on independent graphs**
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→ can be understood as **hopping on independent graphs**

• works for certain **dissipative theories**, e.g. for particle losses:

\[
\dot{\rho}(t) = -i[H_B, \rho(t)] - \mathcal{L}[\rho(t)]
\]
describes loss of particles

\[
\Rightarrow \dot{\alpha}_j(t) = \dot{\alpha}_j^{\text{Ham}}(t) - \dot{\alpha}_j^{\text{diss}}(t) \leq \dot{\alpha}_j^{\text{Ham}}(t)
\]

\[
\geq 0
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- idea extendible to **continuum theories**:
  
either continuum limit, or continuous differential inequalities for \(\alpha(x, t)\)
Summary

- We have studied the propagation of interacting bosons
- We have found a finite propagation speed for any excitation into the initially unoccupied region
- Propagation speed only depends on coupling strength
- Extends to Bose-Fermi mixtures, dissipative models, continuum theories

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