The 2D AKLT state is universal for measurement-based quantum computation

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Outline of the talk

**Introduction**

- measurement-based quantum computation (MQC)
- 2D AKLT state

**Part 1 (Miyake)**

Simulating a quantum circuit model (arXiv:1009.3491)

**Part 2 (Wei)**

Transforming into a 2D cluster state (arXiv:1009.2840)
Measurement-based quantum computation

universal QC model by

- many-particle entanglement
- single-particle measurements
- communication of outcomes

significant questions

- characterization of entanglement which enables one to simulate universal quantum computation (BQP)?
- practical implementation? (large-scale entanglement)

[MQC on a 2D cluster state:
Raussendorf & Briegel, PRL '01; Raussendorf, Browne, Briegel, PRA '03 ]
2D AKLT state

\[ |g\rangle = \bigotimes_v P_v^{S=\frac{3}{2}} \bigotimes_e |\text{singlet}\rangle_e \]

- **ground state of antiferromagnetic two-body** Hamiltonian of spin 3/2's on 2D hexagonal lattice

\[ H = J \sum_{(k,k')} \text{nn.} \left[ S_k \cdot S_{k'} + \frac{116}{243} (S_k \cdot S_{k'})^2 + \frac{16}{243} (S_k \cdot S_{k'})^3 \right], \]

\[ P^3 : \text{projector to total spin 3 for every pair} \]

- **valence bond solid state** (to materialize a spin liquid)

- **merits as resource**: a preparation by cooling stability of a gapped ground state

[Affleck, Kennedy, Lieb, Tasaki, PRL '87; CMP '88]
FAQ: where are qubits?

- area law of entanglement
- stay in degenerate ground states (cf. topological feature)

\[ |g\rangle = \bigotimes_{v} P_{v}^{S=\frac{3}{2}} \bigotimes_{e} \frac{1}{\sqrt{2}} \left( |01\rangle - |10\rangle \right)_{e} = \sum_{\alpha=\pm^{3}/2,\pm^{1}/2} \text{tr} \left[ \prod_{v} A[\alpha_{v}] |\alpha_{v}\rangle \right] \]

"symmetric 3-indices tenors"

\[ A[+\frac{3}{2}] \sim 000^{\mu} \]
\[ A[+\frac{1}{2}] \sim \frac{1}{\sqrt{3}} \left( 001 + 010 + 100^{\mu} \right) \]

upshot of QC: scattering process among edge states
Challenge to construct MQC Protocol

- entanglement network to gate-teleport quantum information
  - Raussendorf, Briegel, PRL’01
  - Verstraete, Cirac, PRA’04
  - Childs, Leung, Nielsen PRA’05
  - Gross, Eisert, PRL’07; Gross, Eisert, Schuch, Perez-Garcia, PRA’07 ...

cluster-state has a VBS-like entanglement structure (PEPS)

- steering quantum information in a controllable (quantum-circuit) manner

1+1D quantum circuit = backbone

How to get unitary maps? How to distinguish space and time?
Outline of MQC Protocol

How to get unitary maps and composed them?

1. measurement at every site, depolarizing \textit{randomly} into one of the three axes

\[ \{M^x, M^y, M^z\} \]

\[ M^\mu = \sqrt{\frac{2}{3}}(\frac{3^\mu}{2}\langle \frac{3^\mu}{2} \rangle + \frac{3^\mu}{2}\langle -\frac{3^\mu}{2} \rangle) \]

\[ \sum_{\mu=x,y,z} M^{\mu\dagger} M^\mu = 1 \]

\textbf{matched bond:} \( \mu_k = \mu_{k'} \)

1'. classical side-computation: in a typical configuration of matched bonds, identifying a backbone (which excludes all sites with triple matched bonds)

2. \textbf{deterministic} quantum computation
Ideas behind MQC Protocol

How to get unitary maps and composed them?

- A (mutually-unbiased) pair of standard and complementary measurements ≤ non-matched bond

- "concentration" from 2D (3-way symmetric) correlation = classical statistical correlation (via random sampling) + "more rigid" quantum correlation
Universal gates and space-time structure

classical information at backbone site:
\[ \Upsilon = X^x Z^z \]

time: two bits sent in the same direction
space: two bits sent in opposite directions
(no net asymmetry in directions)
Summary of Part 1

ground state of a realistic 2D condensed matter system (valence bond solid phase) can be harnessed as a resource of measurement-based quantum computation.

new perspective to traditionally-intractable complexity of 2D quantum systems

Second part:
Converting AKLT state to cluster state

Tzu-Chieh Wei, Ian Affleck and Robert Raussendorf

Ref: arXiv: 1009.2840
Spin-3/2 AKLT state on honeycomb

- Each site contains three virtual qubits
- Two virtual qubits on an edge form a singlet
- Projection \((P_{S,v})\) onto symmetric subspace of 3 qubits at each site

\[ P_{S,v} = |000\rangle \langle 000| + |111\rangle \langle 111| + |W\rangle \langle W| + |\bar{W}\rangle \langle \bar{W}| \]

- Unique ground state of

\[ H = \sum_{\text{edge } \langle i,j \rangle} \hat{P}^{(S=3)}_{i,j} = \sum_{\text{edge } \langle i,j \rangle} \left[ \vec{S}_i \cdot \vec{S}_j + \frac{116}{243} (\vec{S}_i \cdot \vec{S}_j)^2 + \frac{16}{243} (\vec{S}_i \cdot \vec{S}_j)^3 + \frac{55}{108} \right] \]
The POVM in terms of virtual qubits

- Three elements satisfy:

\[ F_{v,x}^\dagger F_{v,x} + F_{v,y}^\dagger F_{v,y} + F_{v,z}^\dagger F_{v,z} = P_{v,sym} \]

\[ F_{v,z} = \sqrt{\frac{2}{3}} (|000\rangle\langle000| + |111\rangle\langle111|) \]

\[ F_{v,x} = \sqrt{\frac{2}{3}} (|++++\rangle\langle++++| + |------\rangle\langle------|) \]

\[ F_{v,y} = \sqrt{\frac{2}{3}} (|i,i,i\rangle\langle i,i,i| + |−i,−i,−i\rangle\langle−i,−i,−i|) \]

\[ a_v = \{x,y,z\} \in A \forall v \]

- In terms of spin-3/2, \( F_{v,a} \) projects onto \( S_a = \pm 3/2 \) subspace

\[ Z|0/1\rangle \equiv \pm|0/1\rangle \]

\[ |\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2} \]

\[ |\pm i\rangle \equiv (|0\rangle \pm i|1\rangle)/\sqrt{2} \]

- POVM outcome (x,y, or z) is random: \( a_v = \{x,y,z\} \in A \) for all sites v

- Post-POVM state becomes

\[ |\Psi(A)\rangle = \bigotimes_{v \in V(\mathcal{L})} F_{v,a_v} |\Phi_{AKLT}\rangle = \bigotimes_{v \in V(\mathcal{L})} F_{v,a_v} \bigotimes_{e \in E(\mathcal{L})} |\phi\rangle_e \]

\( \text{singlets} \)
First result: the post-POVM state is an encoded graph state

\[ |\Psi(\mathcal{A})\rangle = \bigotimes_{v \in V(\mathcal{L})} F_{v,a_v} |\Phi_{\text{AKLT}}\rangle = \bigotimes_{v \in V(\mathcal{L})} F_{v,a_v} \bigotimes_{e \in E(\mathcal{L})} |\phi\rangle_e \]
Graph states: stabilizer formalism

- Stabilizer generators:
  \[ K_v = X_v \otimes Z_u \quad \text{for } u \in \text{Nb}(v) \]

- On arbitrary graph: graph states
  \[ K_v |G\rangle = |G\rangle \]

- On regular lattice: Cluster states
  \[ |C\rangle = |C\rangle \]

Note: \( X \equiv \sigma_x, \ Y \equiv \sigma_y, \ Z \equiv \sigma_z \)
First result: the post-POVM state is an encoded graph state

$$|\Psi(\mathcal{A})\rangle = \bigotimes_{v \in \mathcal{V}(\mathcal{L})} F_{v,a_v} |\Phi_{\text{AKLT}}\rangle = \bigotimes_{v \in \mathcal{V}(\mathcal{L})} F_{v,a_v} \bigotimes_{e \in \mathcal{E}(\mathcal{L})} |\phi\rangle_e$$

What is the graph? Ans. Determined by two rules

- Rule 1: merge neighboring sites of same POVM outcome

- Rule 2: modulo-2 inter-domain edges

Encoding:

- For $m=\text{even}$:
  $$|0\rangle \equiv |(000), (111), \ldots\rangle \equiv |\frac{3}{2}, -\frac{3}{2}, \ldots\rangle$$
  $$|1\rangle \equiv |(111), (000), \ldots\rangle \equiv |\frac{-3}{2}, \frac{3}{2}, \ldots\rangle$$

- These can be reduced to single site

Diagram showing the process of merging sites to form domains and the encoding process based on $m$ (odd/even).
Example

- For convenience use brick-wall structure to represent honeycomb
- Use open boundary condition (terminated by spin-1/2’s, not drawn)

POVM outcomes: $x$ (blue), $y$ (green) or $z$ (red)

Rule 1
merge sites
Example cont’d

Rule 2

Mod-2 on edges
Second result: can convert typical graph states to cluster states

- Typical graphs are in percolated phase (with macroscopic # of vertices $|V|$, edges $|E|$, independent loops or Betti # $B$)

  - Honeycomb: deg=3
    \[ |E| = 1.5|V|, \quad B = 0.5|V| \]

  - Typical graphs: deg=3.52
    \[ |E| = 1.76|V|, \quad B = 0.76|V| \]

  - Square lattice: deg=4
    \[ |E| = 2|V|, \quad B = |V| \]

- Can identify a suitable subgraph and trim it down (by Pauli meas.) to a square lattice:
Summary

- We showed that the 2D AKLT state on the honeycomb lattice is universal for measurement-based quantum computation

I. First approach: constructed a scheme for measurement-based quantum computation (single + two-qubit gates)

  ✓ arXiv: 1009.3491 by Miyake

II. Second approach: showed it can be locally converted to a cluster state

  ✓ arXiv: 1009.2840 by Wei, Affleck & Raussendorf