Tsirelson's Problem and Kirchberg's Conjecture

Tobias Fritz

tobias.fritz@icfo.es

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Abstract

This document is an extended abstract of [Fri10b]. See [JNP+10] for closely related results obtained by different methods.

Quantum correlations in composite quantum systems. In the study of quantum entanglement and quantum correlations, one usually assumes that the state space of a composite quantum system is a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, so that the correlations take on the form

$$P(a, b \,|\, x, y) = \langle \psi, (A_x^a \otimes B_y^b)\psi \rangle. \tag{1}$$

with POVM observables A_a^x and B_b^y. However, how is this justified from physical principles? Can we really be sure that this tensor product assumption is appropriate?

One possible alternative assumption might be to say that a composite system is defined in terms of a joint Hilbert space $\mathcal{H}$ together with, for each site, a set of local observables on $\mathcal{H}$, such that each observable located on the first site commutes with each observable located on the second site; in physical terms, this means that the observables located at different sites are compatible, and can in particular be measured jointly. This is the "commutativity assumption". In the case of finite-dimensional case, this is effectively equivalent to the tensor product assumption [SW08]; however, for infinite-dimensional systems, this fails. So it should always be kept in mind all conjectural statements we make about quantum correlations are only relevant in the case of infinite-dimensional systems.

There is one fundamental reason to consider the commutativity assumption as an alternative to the tensor product assumption. It is our point of view that the operation of forming a composite system from its subsystems should not be a fundamental structure in a physical theory. The point is that nature presents us with a huge quantum system which we observe and conduct experiments with, and in some ways this total system behaves as if it were composed of smaller parts. Hence it seems that the correct question would be "When does a physical system behave like it were composed of smaller parts?" rather than "How do physical systems compose to composite systems?". Note that this is in stark contrast to many other approaches to the foundations of quantum theory, in which the operation of forming a composite system from subsystems is a fundamental structure: e.g. categorical quantum mechanics [Coe10] or certain approaches of reconstructing quantum mechanics from axioms on the probabilistic structure of the theory [Har01], [MM10]. From our point of view, the tensor product operation should not be a fundamental structure of quantum theory, and hence we see the need to consider other structures pertaining to physical systems which potentially make the systems behave like they were composed of subsystems.

Given this motivation, we study what happens to the set of quantum correlations upon relaxing the tensor product structure in (1) to the commutativity assumption. Then the correlations take on the form

$$P(a, b \,|\, x, y) = \langle \psi, A_x^a B_y^b \psi \rangle \tag{2}$$

for which the commutativity assumption $[A_a^x, B_b^y] = 0$ is relevant for ensuring that the imaginary part of this expectation value vanishes.

Tsirelson's problem. The two assumptions 1 and 2 each give rise to a set of quantum correlations as a subset of all no-signaling conditional probability distributions. Calling these sets $\mathcal{Q}_\otimes$ and $\mathcal{Q}_c$, respectively, we arrive at:

> **Tsirelson's problem.** Is $\mathcal{Q}_\otimes = \mathcal{Q}_c$ or $\mathcal{Q}_\otimes \neq \mathcal{Q}_c$?

Of course, the answer to this question may in principle depend on the specific Bell scenario under consideration. We therefore consider the hypothesis:

> **TP conjecture**: $\mathcal{Q}_\otimes = \mathcal{Q}_c$ holds in all bipartite Bell-test scenarios with fixed finite number of observables per party and fixed finite number of outcomes per observable.

At present, the TP conjecture is wide open, and nothing is known besides some relatively simple observations. Firstly, $\mathcal{Q}_\otimes \subseteq \mathcal{Q}_c$ holds in all scenarios, since observables acting on separate tensor factors automatically commute, so that the tensor product assumption implies the commutativity assumption.

What would be the implications of an answer to TP conjecture? Clearly, a positive answer would be a nice justification for assuming quantum correlations to have the form 1; even if the analogous question in the multipartite case would still be open. A negative answer in terms of some correlations which are of the form 2 but not of the form 1 however would probably have a large impact since it would mean that much of the research done since the inception of quantum information theory until today would actually *not be applicable to these quantum correlations*! Also, it would certainly raise many more questions: could these correlations be physically realistic, despite results like those mentioned in [RS10]? If so, would they also be experimentally accessible? Would they be more useful for quantum communication and computation than those of the form 1? Furthermore, it would provide a physically intuitive context in which infinite-dimensional Hilbert spaces of states cannot always be approximated by finite-dimensional ones.

Note also that many bounds on the set of quantum correlations, for example the Navascués–Pironio–Acín hierarchy of semidefinite programs [NPA08], actually bound $\mathcal{Q}_c$ and not $\mathcal{Q}_\otimes$.

Kirchberg's QWEP conjecture. The dichotomy between the tensor product assumption and the commutativity assumption also prevails in the theory of tensor products of C^*-algebras (see e.g. [KR97]). Given C^*-algebras A and B, we may think of them as observable algebras [Lan09] of some physical system, representing all possible superselection sectors at once. Then there are (at least) two C^*-algebraic tensor products which are candidate C^*-algebras for representing the composite physical system,

$$A \otimes_{\min} B \quad \text{and} \quad A \otimes_{\max} B\,, \tag{3}$$

In the first case, the superselection sectors of the joint system are exactly the tensor products of superselection sectors of A and B, while in the second case there exist additional superselection sectors in which the algebras commute, but the state space does not split as a tensor product. In general, the two algebras in 3 are different! Determining whether 3 coincide for a particular pair of C^*-algebras is often a very difficult problem. Kirchberg [Kir93] (see [Oza04] for a more recent review) has proposed the following as an open problem:

> **QWEP conjecture:** $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2)$.

Here, $\mathbb{F}_2$ stands for the free group on two generators, while $C^*(\mathbb{F}_2)$ is the corresponding maximal group C^*-algebra [KR97]. The QWEP conjecture is known to be equivalent to many open problems in C^*-algebra theory related to finite-dimensional approximability, and also to the notorious Connes embedding conjecture for von Neumann algebras [Cap10]; one of the former reformulations is also responsible for the acronym "QWEP" [Oza04, p.7 / 3.19].

Quantum correlations and group C^*-algebras. It is shown in our paper [Fri10b]—and independently in [JNP$^+$10]—that the following implication holds:

$$\boxed{\text{QWEP conjecture} \Longrightarrow \text{TP conjecture}} \tag{4}$$

Note that it is not known whether the converse implication is also true; we will soon get to saying something about how to formulate a variant of Tsirelson's problem *equivalent* to the QWEP conjecture. Results of this type are important in that they provide a physical interpretation of the latter. Thereby it becomes possible to try to attack this purely mathematical problem using physical intuition and physical principles: for example, one might try to look for a counterexample to TP conjecture in terms of correlations of the form 2 which provably violate the physical principle of Information Causality [PPK$^+$09]; and by 4, this would then automatically yield a disproof of the QWEP conjecture.

So how does the correspondence 4 come about? The basic idea is very simple and consists in replacing the outcome labels of an m-outcome projective observable by the mth roots of unity $e^{\frac{2\pi ij}{m}}$, so that the observable becomes a unitary operator of order m. Likewise, a collection of k projective observables with m outcomes is equivalent to a collection of k unitaries of order m. The latter data, in turn, is nothing but a representation of the group C^*-algebra $C^*(\mathbb{Z}_m * \ldots * \mathbb{Z}_m)$. Now it should be plausible that the conjectures TP and QWEP are intimately related.

Moreover, we believe that these group C^*-algebras generally provide a useful and relevant framework for the classification of quantum correlations. For example, it is our impression that the "semidefinite hierarchy" [NPA08] is secretly based upon [Fri10b, prop. 3.4]. Also, a suitable choice of language is always crucial for gaining deeper understanding of a problem. So besides presenting and proving our results, we hope to convince the reader that the language of C^*-algebra tensor products is a suitable framework for Tsirelson's problem, and for the study of quantum correlations in general[1]. For related approaches based on the languages of *operator systems* and *operator spaces*, see [SW08], [JNP$^+$10] and [JPPG$^+$10].

Variants of Tsirelson's problem. Besides the sets of nonlocal quantum correlations, there are many other things one can study in order to understand both the power of the quantum-mechanical formalism and its limitations. We do so by defining two extensions of the concept of quantum correlations, both motivated by our C^*-algebraic picture, and formulate Tisrelson's problem for these.

The first extensions of the concept of quantum correlations is the notion of *spatiotemporal quantum correlations*. Here, it is assumed that the measurements of both parties are projective and do not destroy the system, so that they can be applied in temporal succession. Since any local measurement necessarily decreases the entanglement contained in the shared bipartite state, it may be surprising that spatiotemporal correlations can nevertheless be stronger than ordinary spatial ones, as [Fri10b, ex. 4.5] demonstrates. The QWEP conjecture also implies a positive answer to the spatiotemporal variant of Tsirelson's problem.

The second extension of the concept of quantum correlations is defined in terms of *steering*. As originally formulated by Schrödinger [Sch35], this is the phenomenon that the statse collapse due to Alice's measurement changes the state of Bob's system. Our version of steering considers the case where both Alice and Bob steer the system of a third party; one can view this as replacing, in the definition of quantum correlations, the ordinary classical probabilities $P(a,b|x,y)$ by unnormalized density matrices $\rho(a,b|x,y)$. We formulate a version of Tsirelson's problem also in this case and prove it to be equivalent to the QWEP conjecture, for each Bell scenario separately (except CHSH). So in particular, if the steering version of the TP conjecture is correct in one non-CHSH scenario, it is automatically correct in all other scenarios. Surprisingly, it is actually sufficient to only consider the marginals $\rho(a|x)$ and $\rho(b|y)$, i.e. no joint measurements have to be considered!

[1] Compare [Fri10a] for an application of the same ideas to the classification of temporal quantum correlations.

References

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Cap10] V. Capraro. A Survey on Connes' Embedding Conjecture, 2010, arXiv:1003.2076.

[Coe10] Bob Coecke. A universe of processes and some of its guises, 2010, arXiv:1009.3786.

[Fri10a] Tobias Fritz. On the existence of quantum representations for two dichotomic measurements. *J. Math. Phys.*, 51(5):052103, 21, 2010.

[Fri10b] Tobias Fritz. Tsirelson's problem and Kirchberg's conjecture, 2010, arXiv:1008.1168v2. Draft version.

[Har01] Lucien Hardy. Quantum theory from five reasonable axioms, 2001, arXiv:quant-ph/0101012.

[JNP+10] Marius Junge, Miguel Navascues, Carlos Palazuelos, David Perez-Garcia, Volkher Scholz, and Reinhard Werner. Connes' embedding problem and Tsirelson's problem, 2010, arXiv:1008.1142.

[JPPG+10] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva, and M. M. Wolf. Operator Space Theory: A Natural Framework for Bell Inequalities. *Phys. Rev. Lett.*, 104(17):170405, Apr 2010.

[Kir93] Eberhard Kirchberg. On nonsemisplit extensions, tensor products and exactness of group C^*-algebras. *Invent. Math.*, 112(3):449–489, 1993.

[KR97] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[Lan09] Nicolaas P. Landsman. Algebraic Quantum Mechanics. In Daniel Greenberger, Klaus Hentschel, and Friedel Weinert, editors, *Compendium of Quantum Physics*, pages 6–10. Springer Berlin Heidelberg, 2009.

[MM10] Lluis Masanes and Markus P. Mueller. A derivation of quantum theory from physical requirements, 2010, arXiv:1004.1483.

[NPA08] Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, 2008.

[Oza04] Narutaka Ozawa. About the QWEP conjecture. *Internat. J. Math.*, 15(5):501–530, 2004.

[PPK+09] Marek Pawlowski, Tomasz Paterek, Dagomir Kaszlikowski, Valerio Scarani, Andreas Winter, and Marek Zukowski. Information causality as a physical principle. *Nature*, 461:1101–1104, 2009.

[RS10] Miklós Rédei and Stephen Summers. When are quantum systems operationally independent? *International Journal of Theoretical Physics*, 49:3250–3261, 2010. 10.1007/s10773-009-0010-5.

[Sch35] E. Schrödinger. Discussion of Probability Relations between Separated Systems. In *Proceedings of the Cambridge Philosophical Society*, volume 31 of *Proceedings of the Cambridge Philosophical Society*, pages 555–+, 1935.

[SW08] V. B. Scholz and R. F. Werner. Tsirelson's problem, 2008, arXiv:0812.4305.