Large violation of Bell inequalities with low entanglement

Carlos Palazuelos
(Joint work with M. Junge)

University of Illinois at Urbana-Champaign, USA

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LHVM vs Quantum Mechanics

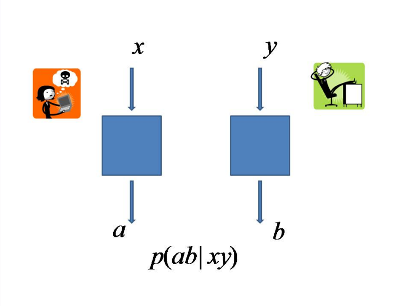

Figure: Alice and Bob measurements. Inputs: x and y, Outputs: a and b. $P(a,b|x,y)$ is the probability of obtaining the pair (a,b) when Alice and Bob measure, respectively, with the input x and y.

We deal with $(P(a,b|x,y))_{x,y=1,\cdots,N}^{a,b=1,\cdots,K} \in \mathbb{R}^J (J = N^2K^2)$.

Probability distributions

Classical probabilities:

$$P = P(a, b|x, y) = \int_{\Omega} P_{\omega}(a|x) Q_{\omega}(b|y) d\mathbb{P}(\omega),$$

a) (Ω, P) is a probability space,
b) $P_{\omega}(a|x) \in \{0, 1\}$ and $\sum_{a=1}^{K} P_{\omega}(a|x) = 1$ for every x, a, ω.

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Quantum probabilities: $P(a,b|x,y) = tr(E_x^a \otimes F_y^b \rho)$,

a) ρ is a density operator acting on $H_1 \otimes H_2$.
b) $E_x^a \geq 0$ for every x, a and $\sum_{a=1}^K E_x^a = \mathbb{1}$ for every x (and analogously for F_y^b).

$$\mathcal{Q} = \{P : P \text{ is quantum}\}$$

"Distance" between quantum and classical probability distributions

Given $M = \{M_{x,y}^{a,b}\}_{x,y=1,a,b=1}^{N,K}$, we define

$$LV(M) = \frac{\sup\{|\langle M, Q\rangle| : Q \in \mathcal{Q}\}}{\sup\{|\langle M, P\rangle| : P \in \mathcal{L}\}} = \frac{\omega^*(M)}{\omega(M)}.$$

Here,

$$\langle M, P\rangle = \sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} P(a, b|x, y).$$

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Note that

$$d(\mathcal{L}, \mathcal{Q}) = f(N, K, d).$$

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1- This is what we need to study $\omega^*(M)$
$\rightsquigarrow$ Difference between *operator algebras* and *operator spaces*

2- Equivalently:

$$M_n \otimes E, \;\; n \geq 1$$

$LV(M) \rightsquigarrow$ classical theory vs non-commutative theory

A nice construction

Let $n \in \mathbb{N}$. Consider $\epsilon_{x,a}^k = \pm 1$ with $x, a, k = 1, \cdots, n$ and define

$$u_x^a = (1, \epsilon_{x,a}^1, \cdots, \epsilon_{x,a}^n) \in \mathbb{R}^{n+1} \quad \text{for every} \quad x, a = 1, \cdots, n.$$

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a) *Bell inequality coefficients*:

$$M_{x,y}^{a,b} = \begin{cases} \frac{1}{n^2}\left(\langle u_x^a, u_y^b \rangle - 1\right) & x, y, a, b = 1, \cdots, n \\ 0 & a = n+1 \text{ or } b = n+1\,. \end{cases}$$

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b) *POVMs measurements*: $\{E_x^a\}_{x,a=1}^{n,n+1}$ in M_{n+1}: For $x = 1, \cdots, n$

$$E_x^a = \begin{cases} |\tilde{u_x^a}\rangle\langle\tilde{u_x^a}| & \text{for } a = 1, \cdots, n \,, \\ 1 - \sum_{a=1}^n E_x^a & \text{for } a = n+1 \,. \end{cases}$$

Here $\tilde{u_x^a} = \frac{1}{\sqrt{nK}} u_x^a$ for certain universal constant K.

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2) For any (diagonal) pure state $|\psi\rangle = \sum_{i=1}^{n+1} \alpha_i |ii\rangle$ we have

$$|\langle M, Q_{|\psi\rangle}\rangle| \succeq \alpha_1 \Big(\sum_{i=2}^{n+1} \alpha_i\Big),$$

where $Q_{|\psi\rangle}(a, b|x, y) = \langle\psi| E_x^a \otimes E_y^b |\psi\rangle$, $x, y, a, b = 1, \cdots, n$.

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Taking $|\varphi_\alpha\rangle = \alpha|11\rangle + \frac{\sqrt{1-\alpha^2}}{\sqrt{n}} \sum_{i=2}^{n+1} |ii\rangle \in \ell_2^{n+1} \otimes \ell_2^{n+1}$ we have

$$LV(M) \succeq \alpha\sqrt{1-\alpha^2}\,\frac{\sqrt{n}}{\log n}.$$

Consequence I:Large violation of Bell Inequalities

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$f(n, n, n) \succeq \frac{\sqrt{n}}{\log n}.$

In the previous talk the authors showed

Theorem (Buhrman, Regev, Scarpa, de Wolf)

$f(\frac{2^n}{n}, n, n) \succeq \frac{n}{(\log n)^2}.$

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Theorem

For any $\delta > 0$ we can find a n-dimensional pure state $|\psi_\delta\rangle$ in a high enough dimension n verifying:

a) $\mathcal{E}(|\varphi\rangle) < \delta$ $\big($resp. $\log_2(n) - \mathcal{E}(|\varphi\rangle) < \delta\big)$,

b)

$$\frac{|\langle M, Q_{|\psi_\delta\rangle}\rangle|}{\sup_{P\in\mathcal{L}} |\langle M, P\rangle|} \succeq \frac{\sqrt{n}}{(\log n)^2},$$

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Even though quantum entanglement is needed to obtain violation of Bell inequalities, the amount of entanglement is essentially irrelevant for large violations.

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b) $\sup\{|\langle M, Q\rangle| : Q \in \mathcal{Q}_{max}\} \leq 1$, where $\mathcal{Q}_{max}$ is the set of quantum probability distributions constructed with the maximally entangled state in *any* dimension.

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1) There exist quantum probability distributions which cannot be written by using *any* maximally entangled state.
2) "Opposite" result to the main one in the previous talk.

THANK YOU VERY MUCH FOR YOUR ATTENTION