Anyons, Twists & Topological Codes

Héctor Bombín
Perimeter Institute
outline

• motivation/introduction
• twists
  • anyon models, symmetries & twists
  • toric code
• topological subsystem color codes
  • twist geometry
  • Clifford operations by code deformation
• conclusions
motivation

• traditional fault-tolerance is not enough to realistically fight decoherence

• topological alternatives (Kitaev):
  • topological quantum computing (TQC)
    • self-protected by energy gap
    • immune to small local distortion
  • topological codes (TC)
    • geometrically local, active error correction
    • error threshold for large size
  • both are anyon-based: exotic statistics in 2D
motivation

- two problems addressed here:
  - TQC: the anyons that are easier to get have no computational power
  - TC: there exist extremely local TCs (2-local measurements in 2D), but no way to compute

- a solution / new tool:
  - twists $\rightarrow$ use anyon symmetries to increase computational power
introduction

- why 2D?

- statistics beyond bosons and fermions
- topological interaction
- appear in systems with topological order (TO) (Wen ’89):
  - gapped, ground state degeneracy depends on topology
introduction

• abelian charge: given charge of constituents, total charge is known

• topological charge can be non-abelian
introduction

- TQC (Kitaev '03, Freedman et al '03)

encode in fusion channels

compute = braid

measure = fuse
introduction

- abelian anyons have no computational power
- twists offer a way to recover non-abelian behavior!
introduction

• quantum error correcting codes protect quantum information using redundancy

• typically this involves encoding in a subspace
introduction

- the code subspace can be defined in terms of commuting observables: **check operators** (CO)

\[ C_i |\psi\rangle = c_i |\psi\rangle \]

- errors typically change CO values → allows to keep track of errors

  CO measurement → error syndrome →

  → compute most probable error
• topological codes (Kitaev '97)

• geometrically **local** check operators = easy to measure

• global undetectable errors = hard to happen
introduction

- # encoded qubits depends on topology (homology)
- flexible: many lattices allowed, transversal gates possible
- **boundaries:** planar geometries
- topological quantum memory (Dennis et al ‘02):
  - measure COs repeatedly
  - under a noise **threshold**, storage time exp in size
  - ideal error correction amounts to compute free energy
- **code deformation:**
  - change topology over time: initialize, compute, measure
• **subsystem** codes (Kribs et al ’05) can also improve locality

• only a subsystem of the code subspace is used

• check operators need not be measured directly → measurements potentially more local (Poulin ’05)
introduction

• **topological subsystem color codes** (TSCC) (Bombin ’09)

• “doubly local”: topology + subsystem

• error syndrome recovery needs 2-local measurements!
• TC and TO are closely related for subspace codes

<table>
<thead>
<tr>
<th>Topological codes</th>
<th>VS</th>
<th>Topological order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Check operators</td>
<td>↔</td>
<td>Hamiltonian terms</td>
</tr>
<tr>
<td>Code subspace</td>
<td>↔</td>
<td>Ground subspace</td>
</tr>
<tr>
<td>Error syndrome</td>
<td>↔</td>
<td>Excitation config</td>
</tr>
</tbody>
</table>

• TSCCs also have an anyonic picture for error syndromes
introduction

- TSCCs do not allow boundaries
  - no natural planar codes
  - code deformation becomes unpractical
- with twists
  - we can build planar TSCCs
  - whole Clifford group by code deformation!
twists

- ingredients of an anyon model:

\[ Q \in \{a, b, \ldots\} \]

\[ Q \times Q' = q_1 + q_2 + \ldots \]

topological charges

fusion rules

braiding rules

\[ |\psi\rangle \longrightarrow \text{?} \]
twists

- ex.: Ising anyons
  - topological charges \( \{1, \sigma, \psi\} \)
  - fusion rules
    \[ \sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1. \]
- the total charge of two distant \( \sigma \)-s is \( 1 \) or \( \psi \):
  - if far apart, global qubit
- fusion space: \( 2n \sigma \)-s \( \rightarrow n \) qubits
twists

• braiding rules: 

• we can describe braiding up to a phase with a **Majorana operator** per $\sigma$

• Majorana operators are self-adjoint $c_i$ with

$$c_j c_k + c_k c_j = 2 \delta_{j,k}$$

• total charge of $j$-th and $j+1$-th $\sigma$-s: 

$$-i c_j c_{j+1}$$

• braiding: 

$$c_j \rightarrow c_{j+1}$$

$$c_{j+1} \rightarrow -c_j$$

• not universal, but we can use distillation (Bravyi '06)
twists

- **anyon symmetry**: charge permutation producing an equivalent anyon model
  \[ q \rightarrow \pi(q) \]

- imagine ‘cutting’ the anyons’ 2D world and gluing it again up to a symmetry
twists

- across the cut, charges change:

\[ q \xrightarrow{\text{cut}} \pi(q) \]

- topologically, the cut location is unphysical.

- endpoints are meaningful: under monodromy they permute charges → twists
twists

- ex.: quantum double of $\mathbb{Z}_2$ (toric code)
- charges: $\{1, e, m, \epsilon\}$
- fusion: $e \times m = \epsilon \quad e \times \epsilon = m \quad m \times \epsilon = e$
  \[ e \times e = m \times m = \epsilon \times \epsilon = 1 \]
- braiding: $e, m \rightarrow$ bosons $\quad \epsilon \rightarrow$ fermion

\[ q \quad q' \quad = - \quad \left\{ q \left\| q' \right. \quad 1 \neq q \neq q' \neq 1 \right. \]

- nontrivial symmetry: $e \leftrightarrow m$
twists

- twists are sinks/sources for fermions:

  ![Diagram of fermion sinks/sources]

- vacuum to vacuum processes...

  ![Diagram of vacuum to vacuum processes]

- ...lead to **topological degeneracy**:

  ![Diagram of topological degeneracy]

\[ S_1 S_2 = -S_2 S_1 \]
twists

- toric code (Kitaev ’97, Wen ‘03):
  - qubits form a square lattice
  - 4-local check operators at plaquettes

\[
C_k := X_k Z_{k+i} X_{k+i+j} Z_{k+j}
\]

\[
C_k |\psi\rangle = |\psi\rangle
\]

- Hamiltonian version:
  \[
  H := - \sum_k C_k
  \]
  - excitations live at plaquettes
**twists**

- **string operators** create/destroy excitations at their endpoints

- two types of strings/excitations: e (light) and m (dark)
twists

- twisting amont to dislocations

- twists can be locally created in PAIRS only
twists

- no twists (or even number) $\rightarrow$ 4 possible charges

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>e</th>
<th>m</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_e$</td>
<td>+1</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$S_m$</td>
<td>+1</td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
</tbody>
</table>

- a twist (or an odd number) $\rightarrow$ 2 possible charges

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_+$</th>
<th>$\sigma_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_e$</td>
<td>+i</td>
<td>-i</td>
</tr>
<tr>
<td>$S_m$</td>
<td>-i</td>
<td>+i</td>
</tr>
</tbody>
</table>
twists

- non-abelian fusion rules!

\[
\sigma_\pm \times \sigma_\pm = 1 + \epsilon \\
\sigma_\pm \times \sigma_\mp = e + m \\
\sigma_\pm \times \epsilon = \sigma_\pm \\
\sigma_\pm \times e = \sigma_\pm \times m = \sigma_\mp
\]

- we recover Ising rules:

\[
\sigma_+ \times \sigma_+ = 1 + \epsilon \\
\sigma_+ \times \epsilon = \sigma \\
\epsilon \times \epsilon = 1
\]
**twists**

- all closed string ops can be expressed in terms of a set of open string ops → Majorana operators

\[ c_j c_k + c_k c_j = 2 \delta_{jk} \]

- braiding is also Ising-like!

\[ c_j \rightarrow c_{j+1} \]

\[ c_{j+1} \rightarrow -c_j \]
twists

toric code
abelian

Ising anyons
non-abelian

twists
TSCC

- The original TSCCs come from 3-valent lattices with **3-colorable** faces (red, green, blue)
- String operators have a color
- Commutation relations of string ops relates them to an anyon model with three nontrivial charges
TSCC

- fusion rules as in toric code
  
  \[ r \times g = b \quad g \times b = r \quad b \times r = g \]
  
  \[ r \times r = g \times g = b \times b = 1 \]

- braiding of different charges as in toric code

- the difference: three fermionic charges

- any permutation of the colors is a symmetry!

- twists are labeled by the elements of \( S_3 \)
TSCC

- faces with an odd number of links brake 3-colorability
- these are twists: two colors are exchanged
- a red twist exchanges green and blue, and so on
• to the i-th twist we attach a string \( \gamma_i \ldots \)

\[
\begin{aligned}
\gamma_{1,3}^r & \quad \gamma_5 & \quad \gamma_7 & \quad \gamma_9 \\
\times \times \times r & \quad \times g & \quad g r & \quad b b & \quad r b & \quad b g & \quad \times b
\end{aligned}
\]

• ...and get self-adjoint string ops \( k_i \),

“colored” Majorana ops

\[
k_i^2 = 1 \text{ and, for } i < j,
\]

\[
k_i k_j = \begin{cases} 
  k_j k_i & \text{if } c_i = \zeta_+(c_j), \\
  -k_i k_j & \text{otherwise.}
\end{cases}
\]
TSCC

- braiding changes the color of twists

\[ \sigma_c(c') \]

- transforming as follows the colored Majorana ops
  \((c_i \text{ is the color of the } i\text{-th twist})\)

\[ k_j \to k_{j+1}, \quad k_{j+1} \to \begin{cases} -k_j & \text{if } c_j = c_{j+1}, \\ ik_j k_{j+1} & \text{if } c_j = \zeta_-(c_{j+1}), \\ -k_j k_{j+1} & \text{otherwise.} \end{cases} \]


TSCC

- for twists of the same color, we are back to Ising anyons
- encoding: 1 qubit = 4 twists of the same color
- we get all single qubit Clifford gates (Bravyi ’06)

\[
\langle k_j k_{j+1} k_{j+2} k_{j+3} \rangle = -1
\]

\[
\hat{X} \equiv -i k_j k_{j+1}
\]

\[
\hat{Z} \equiv -i k_{j+1} k_{j+2}
\]
to get the whole Clifford group, we only need to implement an **entangling gate**

but for two groups of twists of different color:

\[
\begin{align*}
\hat{X}_1 &\rightarrow \hat{X}_1 & \hat{Z}_1 &\rightarrow \hat{X}_2 \hat{Z}_1 \\
\hat{X}_2 &\rightarrow \hat{X}_2 & \hat{Z}_2 &\rightarrow \hat{X}_1 \hat{Z}_2
\end{align*}
\]

and we can always flip the color of a group:
TSCC + twists +
+ code deformation =
Clifford gates
conclusions & questions

- anyon symmetries allow to introduce twists
- twists make anyon models and topological codes computationally more powerful (but how much?)
- toric codes:
  - twists mimic Ising anyons
- topological subsystem color codes:
  - Clifford operations by code deformation
- other/general anyon models?

PRL 105.030403 / arXiv:1004.1838
arXiv:1006.5260