

Finding is as easy as detecting for quantum walks

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Abstract. We solve an open problem of constructing a quantum walk that not only detects but also finds marked vertices in a graph. The number of steps of our quantum walk is quadratically smaller than the classical hitting time of any reversible random walk P on the graph. Our approach is new, simpler and more general than previous ones. We introduce a notion of interpolation between the classical walk P and the absorbing walk P' , whose marked states are absorbing. Then our quantum walk is simply the quantum analogue of this interpolation. Contrary to previous approaches, our results remain valid when the random walk P is not state-transitive, and also in the presence of multiple marked vertices. As a consequence we improve the best known algorithm for spatial search on the 2D-grid.

1 Introduction

Many classical randomized algorithms rely heavily on random walks or Markov chains. Quantum walks are natural generalizations of classical random walks and have similarly found many applications. Ambainis [1] was the first to solve a natural problem – the “element distinctness problem” – using a quantum walk. Following this, many quantum walk algorithms were discovered [2–4].

In spatial search problem the displacement constraints are modeled by an undirected graph G and the desired vertices are marked. Classically, a simple algorithm to find a marked vertex is to repeatedly apply some random walk P on G until one of the marked vertices is reached. The expected running time of such algorithm is known as the hitting time of P .

Quantum walk algorithms for the spatial search problem [5] were studied for hypercube [6] and grid [7, 8] and the notion of hitting time has been carried over to the quantum case in [8–13]. Usually, the quantum hitting time has a quadratic improvement over the classical one. However, until the present paper, several serious restrictions were imposed: a quantum algorithm could only detect the presence of marked vertices [10], but to be able to find them the Markov chain was assumed to be reversible, state-transitive, and possess a unique marked vertex [14, 13]. The existing detection algorithm is quite intuitive and well understood, whereas the finding algorithm requires an elaborate proof whose intuition is not clear. This is due in part to a modification of the quantum walk, so that the resulting walk is not a quantum analogue of a Markov chain anymore.

Whether this quadratic speed-up for finding a marked vertex also holds for any reversible Markov chain and for multiple marked vertices was an open question. In the technical version [15] of the present paper, we answer this question in the positive. We choose another approach by modifying P directly, and by considering the quantum analogue of the modified random walk. Using this construction we obtain more general results while simplifying the proofs of existing results. Moreover, this construction is more intuitive than previous approaches.

2 Context

The problem

We consider an undirected graph G , with an unknown set of marked vertices M . Our goal is to find any of the marked vertices using only evolutions that preserve the locality of G , i.e., to perform a *spatial search* on G [5]. We denote this problem by $\text{FIND}(G)$, and also consider a weaker version $\text{DETECT}(G)$, where only the presence of marked vertices has to be detected (i.e., detect whether M is non-empty). We will add a superscript “ $(=k)$ ”, or “ $(\geq k)$ ”, to denote promise versions of these problems, where the promise is that $|M| = k$, or $|M| \geq k$, whenever it is non-empty.

Classical hitting time

A natural approach to searching on a graph consists of using a random walk, or Markov chain. Let P be an ergodic and reversible Markov chain. The classical hitting time $\text{HT}(P, M)$ of Markov chain P , is defined as the expected number of applications of P required to hit a marked vertex in M when starting from the stationary distribution π of P . This can be used to design a randomized algorithm for DETECT and FIND based on the corresponding random walk.

Proposition 1. $\text{FIND}^{(\geq k)}(G)$ can be solved with high probability and randomized complexity of order $T = \max_{|M'|=k} \text{HT}(P, M')$.

Quantum hitting time

Quantum walks were successfully used by Szegedy for detecting the presence of marked vertices quadratically faster than using P .

Theorem 1 (Szegedy, [10]). $\text{DETECT}^{(\geq k)}(G)$ can be solved with high probability and quantum complexity of order $T = \max_{|M'|=k} \sqrt{\text{HT}(P, M')}$.

Nonetheless, very little was known about the problem of finding a marked vertex, except in the restricted case where P is state-transitive and there is a unique marked vertex (i.e., $|M| = 1$).

Theorem 2 ([14, 13]). Assume that P is state-transitive. $\text{FIND}^{(1)}(G)$ can be solved with high probability and quantum complexity of order $T = \sqrt{\text{HT}(P, M)}$.

This theorem improves and extends previous ad-hoc results for the hypercube [6] and the grid [7, 8]. However, before the present article, no result was known for a Markov chain that is not state-transitive.

3 Main result

Let $p_M = \sum_{x \in M} \pi_x$ be the probability to pick a marked vertex from the stationary distribution π of P . Our main result is most simply expressed in the case where an approximation p_M^* of p_M is known.

Theorem 3. *Let p_M^* be such that $|p_M^* - p_M| \leq p_M/3$. Then, $\text{FIND}^{(\geq k)}(G)$ can be solved with high probability and quantum complexity of order $T = \max_{|M'|=k} \sqrt{\text{HT}(P, M')}$.*

Note that the assumption that an approximation of p_M is known is very similar to the requirement that the number of marked elements in the original version of Grover's algorithm is known. It is shown in the technical article [15] how to deal with the case of an unknown p_M . If not for this restriction, this theorem extends the result of Theorem 1 to the FIND problem. Unlike in Theorem 2, it imposes no limitation on the Markov chain (except being ergodic and reversible) and the number of marked vertices. It therefore provides a generic quadratic speed-up on the classical random walk algorithm in Proposition 1.

Algorithm

Here is a high-level description of the algorithm that leads to Theorem 3. We use Szegedy's technique [10] to design a quantum analogue of a Markov chain. While previous algorithms for this problem proposed to use the quantum analogue of P itself, or of an absorbing version P' , where marked vertices are probability sinks, the novelty of our approach is to consider a Markov chain which is an interpolation at the classical level i.e., $P(s) = (1-s)P + sP'$ for $0 \leq s \leq 1$. Then, we consider the quantum analogue $W(P(s))$, of this interpolation. This is similar to the concept of adiabatic evolution and indeed leads to a surprising connection between the notions of adiabatic condition and quantum hitting time [16]. The stationary distribution $\pi(s)$ of $P(s)$ interpolates between the stationary distribution π of P , and its projection onto the marked vertices. Let X be the state space of P . Then from Szegedy's construction, the quantum analogue $W(P(s))$ will have as unique 1-eigenvector the following state:

$$|\pi(s)\rangle|0\rangle = \sum_{x \in X} \sqrt{\pi_x(s)}|x\rangle|0\rangle = \cos \theta(s)|U\rangle|0\rangle + \sin \theta(s)|M\rangle|0\rangle, \quad (1)$$

where $\theta(s) = \arcsin \sqrt{p_M/[1 - s(1 - p_M)]}$ and $|M\rangle, |U\rangle$ are the normalized projections of $|\pi\rangle = \sum_{x \in X} \sqrt{\pi_x}|x\rangle$ onto marked and unmarked vertices, respectively. Hence, the interpolation coefficient s can be used to tune the overlap between the 1-eigenvector of the quantum walk and its projection $|U\rangle$ onto marked vertices. For $s^* = \frac{1-2p_M}{1-p_M}$, the overlap with marked and unmarked vertices is balanced. The algorithm then simply consists of measuring the state $|\pi(0)\rangle|0\rangle$ in the eigenbasis of $W(P(s^*))$, which can be done by quantum phase estimation [17, 18]. This projects the state onto $|\pi(s^*)\rangle|0\rangle$ with probability close to 1/2. Measuring the first register of this state in the computational basis then yields a marked element with probability 1/2, which therefore solves the problem. It remains to prove that the cost of the quantum phase estimation is given by the square root of the hitting time $\text{HT}(P, M)$, which is done in the technical article [15].

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