Topological quantum order: stability under local perturbations

Sergey Bravyi, IBM
Matthew Hastings, Microsoft
Spiros Michalakis, Caltech

QIP 2011, Singapore,
January 10-14, 2011

arXiv:1001.4363
Why the existence of topologically ordered phases of matter is surprising?

1. Ground states of TQO models are highly entangled. This entanglement cannot be accounted only by local correlations. Non-local entanglement in macroscopic systems is extremely fragile.

2. How can nature prepare these highly entangled states without having a large-scale quantum computer?

3. Many models of TQO require multi-spin interactions which are not very realistic.
Quantum spin lattices

Finite-dimensional quantum spins live at sites.

Hamiltonian of the ideal model:

\[ H_0 = \sum_{B \subseteq \Lambda} Q_B \]

\[ \| Q_B \| \leq 1 \]

To what extent ground state properties of \( H_0 \) are sensitive to addition of weak local perturbations?

\[ H_0 \rightarrow H_0 + \epsilon V, \quad V = \sum_{B \subseteq \Lambda} V_B, \quad \| V_B \| \leq 1. \]
Gap stability

The ground state $\Psi(0)$ is a $g$-dimensional subspace. The ground subspace $\Psi(\epsilon)$ includes $g$ smallest eigenvalues of $H_0 + \epsilon V$. Well-defined as long as $\Delta(\epsilon) > 0$.

Main goal: find sufficient conditions under which $H_0$ has a non-zero stability radius $\epsilon_0$, that is, the gap $\Delta(\epsilon)$ has a constant ($L$-independent) lower bound on the interval $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$. 
Exact quasi-adiabatic continuation theorem
(Hastings 2005, 2010, Osborne 2007)

Suppose the spectral gap $\Delta(\lambda)$ has a constant lower bound for $\lambda \in [0, \epsilon]$. Then $\Psi(0)$ and $\Psi(\epsilon)$ can be mapped to each other by some unitary operator $U$,

$$
\Psi(\epsilon) = U \cdot \Psi(0)
$$

and $U$ can be implemented in time $t = O(1)$ as $L \rightarrow \infty$.

$$
U = T \cdot \exp \left( i \int_{0}^{t} ds \, G_\epsilon(s) \right)
$$

Here $G_\epsilon(s)$ is a local (approximately) Hamiltonian with bounded strength of interactions.

The states $\Psi(0)$ and $\Psi(\epsilon)$ are in the same “topological phase” if $\Psi(\epsilon) = U \cdot \Psi(0)$ and $U$ can be implemented by evolution under a local Hamiltonian in time $O(1)$. 
Can it happen that all ground states of local Hamiltonians are in the same phase?

Topological trivial phase = product states

States with long-range bipartite entanglement would be in a non-trivial phase

Generating a singlet between remote qubits by local unitary dynamics can take time of order $L$ (Lieb-Robinson bound).

However, such states cannot appear as ground states of local Hamiltonians.
Topological quantum order: a pattern of long-range multipartite entanglement that can be present in the ground states of local Hamiltonians.

Toric code state (Kitaev 97)

Qubits live on links.

\[ |\Psi_{tc}\rangle \sim \sum_{\text{cycles}} |C\rangle \]

\[ H_0 = - \sum_{\text{stars}} A_s - \sum_{\text{plaquettes}} B_p \]

Star operators:

\[ A_s = \begin{array}{c} Z \ \ \ Z \\ Z \ \ \ Z \end{array} \]

Plaquette operators:

\[ B_p = \begin{array}{c} X \ \ X \\ X \ \ X \end{array} \]
Topological quantum order: a pattern of long-range multipartite entanglement that can be present in the ground states of local Hamiltonians.

\[ |C\rangle \]

**Toric code state (Kitaev 97)**

Qubits live on links.

\[ |\Psi_{tc}\rangle \sim \sum_{\text{cycles}} |C\rangle \]

**Theorem** (S.B., Hastings, and Verstraete 2006):
The toric code state on a lattice of size \( L \) cannot be generated from the product state by local unitary dynamics in time \( o(L) \).

\[ \Rightarrow \Psi_{tc} \text{ and the product state are in different phases.} \]
Previous work: gap stability for the toric code for several special perturbations:

Trebst, Werner, Troyer, Shtengel, Nayak (2007)
Magnetic field diagonal in the $Z$-basis
Reduction to the 2D transverse field Ising model

Vidal, Thomale, Schmidt, Dusuel (2009)
Magnetic field diagonal in the $Y$-basis
Reduction to the Xu-Moore model; use of self-duality

Klich (2009)
Generic perturbations; non-degenerate ground state
Generic perturbations diagonal in the $Z$-basis
Cluster expansions for the partition function
New results: sufficient gap stability conditions for a large class of ideal models $H_0$ and generic perturbations.

**TQO-1:** Ground subspace of $H_0$ is a quantum code with a macroscopic distance.

**TQO-2:** Consistency between the global and the local ground subspaces of $H_0$
(formal definition will appear later)

TQO-1 is only a property of the ground state

TQO-2 is a property of the Hamiltonian

We shall prove that TQO-1,2 together are sufficient for stability under generic local perturbations
Unperturbed Hamiltonian:

\[ H_0 = \sum_{B \subseteq \Lambda} Q_B \]

\( Q_B \) is a Hermitian operator acting only on a cluster \( B \)

Only 2 \( \times \) 2 clusters

\( Q_B \) must be a projector: \( Q_B^2 = Q_B \)

Projectors must pairwise commute: \( Q_B Q_C = Q_C Q_B \)

Ground states of \( H_0 \) are zero-eigenvectors of every projector \( Q_B \)
To summarize, we need three properties of the ideal model:

- Spatially local
- Frustration free
- Term-wise commuting

Several extra conditions related to TQO will be introduced later...
Examples:

- The toric codes and the surface codes
- Topological color codes
- Quantum double models
- String-net models
- Any of the above models with excitations

\[ H_0 = \sum_B Q_B \]

Kitaev 97
Bombin and Martin-Delgado 06
Levin and Wen 05

Commutativity guarantees that \( H_0 \) has constant spectral gap (\( \Delta \geq 1 \)) above the ground state!
Generic perturbations:

\[ V = \sum_{B \subseteq \Lambda} V_B \]

\( V_B \) is a Hermitian operator acting only on a cluster \( B \)

Exponential decay of interactions:

For clusters of size \( r \times r \)

\[ \max_B \| V_{r,B} \| \leq \exp(-\mu r) \]

\[ \mu = \text{decay rate} \]
\[ H_0 = \sum_B Q_B \]

Global ground subspace:

\[ P = \{ |\psi\rangle : Q_B |\psi\rangle = 0 \quad \forall B \} \]

\[ P \equiv \Psi(0) \]

Local ground subspace:

\[ P_M = \{ |\psi\rangle : Q_B |\psi\rangle = 0 \quad \forall B \subseteq M \} \]

zero eigenvectors only for projectors \( Q_B \)

whose support is contained in \( M \).
TQO-1 (macroscopic distance):

Global ground states cannot be distinguished locally:

\[ \langle \psi | O_M | \psi \rangle = \langle \phi | O_M | \phi \rangle \quad \forall \psi, \phi \in P \]

for any operator \( O_M \) acting on \( M \)

Holds for all \( M \)'s of size \( \leq L^\alpha \).

---

TQO-2 (global-local consistency):

\( O_M P = 0 \) implies \( O_M P_{M^+} = 0 \)

for any operator \( O_M \) acting on \( M \)

Holds for all \( M \)'s of size \( \leq L^\alpha \).
What is the meaning of TQO-2 for stabilizer codes?

Global ground states are invariant under the action of a stabilizer group

$$S = \langle G_1, \ldots, G_m \rangle$$

Generators $G_a$ are pairwise commuting Pauli operators.

Each generator is supported on a $2 \times 2$ block $B$.

$$H_0 = \sum_B Q_B$$

$Q_B$ penalizes states that violate at least one generator $G_a$ supported inside $B$. 
What is the meaning of TQO-2 for stabilizer codes?

TQO-2 (global-local consistency):
\[ O_M P = 0 \text{ implies } O_M P_{M^+} = 0 \]

Holds for all \( M \)'s of size \( \leq L^\alpha \).

Lemma. A stabilizer code Hamiltonian \( H_0 \) obeys TQO-2 iff any stabilizer \( S \in \mathcal{S} \) supported on \( M \) can be written using only generators supported on \( M^+ \).

Toric code: if a loop operator has support inside \( M \), it is a product of plaquette operators supported inside \( M \).
To summarize, we need five properties of the ideal model $H_0$:

- Spatially local
- Frustration free
- Term-wise commuting
- Macroscopic distance (TQO-1)
- Local-global consistency (TQO-2)

The perturbation $V$ involves exponentially decaying interactions with strength $J$ and decay rate $\mu > 0$. 
Main theorem:

There exists a constant $c = c(\mu, \alpha)$ such that for all large enough $L$ and for all $\epsilon > 0$ the spectrum of $H_0 + \epsilon V$ is contained (up to an overall energy shift) in the union of intervals

$$\bigcup_k I_k$$

- $k$ runs over eigenvalues of $H_0$
- Interval $I_k$ is centered at $k$
- $|I_k| = \epsilon ck$ for $k > 0$
- $|I_0| = \epsilon c \cdot \exp(-\sqrt{L})$

Corollary: the spectral gap around $I_k$ is at least $1/2$ for all $\epsilon \leq \epsilon_k = (1 + 2k)^{-1}(2c)^{-1}$. 
1. The bound on the stability radius does not depend on the dimension of the local Hilbert spaces.

2. The overall energy shift may be a function of $L$ and $\epsilon$.

3. Conditions TQO-1,2 can be efficiently checked for any stabilizer code Hamiltonian.

4. The theorem applies to systems with symmetries. A system has a symmetry group $\mathcal{G}$ iff all local terms in $H_0$ and $V$ commute with $\mathcal{G}$. Conditions TQO-1,2 must be obeyed only for operators $O_M$ commuting with $\mathcal{G}$.

**Symmetry protected topological order:** non-trivial topological phases may exist even in 1D systems.

X. Chen, Z.-C. Gu, and X.-G. Wen

5. Conditions TQO-1,2 are well-defined for classical $H_0$.

TQO-1: All ground states are locally indistinguishable. Hence unique ground state.

TQO-2: If some spin $\sigma_u$ deviates from its ground state value, at least one interaction touching $\sigma_u$ is violated.

Analogous to the Peierls condition in the stability theory for quantum perturbations of classical Hamiltonians (Datta, Frölich, Rey-Bellet 1997)
Why the ground state energy splitting is exp. small?

Exact quasi-adiabatic continuation theorem implies

\[ \psi_\alpha(1) = U \cdot \psi_\alpha(0), \quad \alpha = 1, \ldots, g, \]

where $U$ describes unitary evolution under (approximately) local Hamiltonian for time $O(1)$.

\[ E_\alpha(1) = \langle \psi_\alpha(1) | H_0 + \epsilon V | \psi_\alpha(1) \rangle = \langle \psi_\alpha(0) | \tilde{H} | \psi_\alpha(0) \rangle, \]

\[ \tilde{H} \equiv U^\dagger (H_0 + \epsilon V) U. \]

Lieb-Robinson bound implies that $\tilde{H}$ is a sum of (approximately) local interactions. Hence $\tilde{H}$ cannot distinguish orthogonal ground states $\psi_\alpha(0)$.

Hence all $E_\alpha(1)$ are (approximately) the same.
Sketch of the proof

Generic perturbations

Block-diagonal perturbations

Stability theorem

Techniques:

- Hamiltonian flow equations
- Lieb-Robinson bounds
- Quasi-adiabatic continuation

Relative bounds on $V$
Def. A perturbation $V$ is relatively bounded by $H_0$ with a constant $b$ iff

$$\|V \psi\| \leq b \|H_0 \psi\|$$

for all vectors $\psi$.

Lemma. The spectrum of $H_0 + V$ is contained in the union of intervals

$$I_k = [(1 - b)k, (1 + b)k]$$

where $k$ runs over eigenvalues of $H_0$.

Applying the lemma to $H_0 + \epsilon V$ we get the desired energy bands $I_k$ as long as $b \cdot |\epsilon| < 1$. Hence we need a bound $b = O(1)$. 
Def. A perturbation $V$ is called locally block-diagonal iff it is a sum of local operators preserving the global ground subspace of $H_0$, that is,

$$V = \sum_{B \subseteq \Lambda} V_B$$

$$V_B \cdot P \subseteq P$$

Macroscopic distance implies that $V_B$ acts trivially on the ground states. Perform an overall energy shift to achieve

$$V_B \cdot P = 0$$

Lemma: A locally block-diagonal perturbation satisfying $V_B \cdot P = 0$ is relatively bounded by $H_0$ with a constant $b = b(\mu)$. 
Block-diagonality \( \Rightarrow \) relative boundness (rough idea)

Decompose the entire Hilbert space into sectors labeled by configurations of excitations.

Simplest case: excitation is a \( 2 \times 2 \) square \( B \) such that \( Q_B = 1 \) instead of \( Q_B = 0 \).
Block-diagonality $\Rightarrow$ relative boundness (rough idea)

\[ V = \sum_{B \subseteq \Lambda} V_B, \quad V_B \cdot P = 0 \]

TQO-2 implies $V_B \cdot P_{B^+} = 0$

Hence $V_B$ annihilates any sector that contains no excitations near $B$.

Assume for simplicity that $\psi$ belongs to a sector with $k$ excitations. Then there are only $O(k)$ terms $V_B$ such that $V_B \psi \neq 0$.

\[ \|V \psi\| \sim k \quad \text{and} \quad \|H_0 \psi\| = k \]

Hence $V$ is relatively bounded by $H_0$ with a constant $b$ of order 1.
Sketch of the proof

Generic perturbations

Block-diagonal perturbations

Stability theorem

Techniques:

- Hamiltonian flow equations
- Lieb-Robinson bounds
- Quasi-adiabatic continuation

Relative bounds on $V$
Suppose $V$ is not block-diagonal. We shall construct a unitary operator $U$ such that

$$U(H_0 + \epsilon V)U^\dagger = H_0 + \epsilon W + H_{\text{garbage}}$$

$W$ is a locally block-diagonal perturbation with a fast enough decay of interactions. $W$ is relatively bounded by $H_0$ with a constant $b = O(1)$.

$H_{\text{garbage}}$ includes all unwanted terms. Must have exponentially small norm.

Hence $\epsilon V$ changes eigenvalues of $H_0$ by a factor $1 \pm \epsilon b$ and an additive error $\|H_{\text{garbage}}\|$.
How to construct $U$ (Hamiltonian flow equations):

First solve the linearized block-diagonalization problem. $U$ only needs to make the Hamiltonian locally block-diagonal in the first order in $\epsilon$:

$$U(H_0 + \epsilon V + \delta W)U^\dagger = H_0 + c\epsilon^2 V' + \delta'W' + H_{garbage}$$

Here $W, W'$ are locally block-diagonal, $V, V'$ are generic perturbations, $c$ is some constant, and

$$\delta' \leq \delta + O(\epsilon)$$

$$U = \exp(\epsilon S), \quad S^\dagger = -S,$$

$$P^\perp \cdot ([S, H_0 + \delta W] + V) \cdot P = 0$$

Construct $S$ using power series in $\delta$. Use the Lieb-Robinson bound to show that $V'$ and $W''$ decay fast enough.
How to construct $U$ (Hamiltonian flow equations):

First solve the linearized block-diagonalization problem. $U$ only needs to make the Hamiltonian locally block-diagonal in the first order in $\epsilon$:

$$U(H_0 + \epsilon V + \delta W)U^\dagger = H_0 + c \epsilon^2 V' + \delta' W' + H_{garbage}$$

Here $W, W'$ are locally block-diagonal, $V, V'$ are generic perturbations, $c$ is some constant, and

$$\delta' \leq \delta + O(\epsilon)$$

Iterate $m = O(\log L)$ times obtaining

$$\epsilon \rightarrow c \epsilon^2 \rightarrow c^3 \epsilon^4 \rightarrow \ldots \rightarrow \frac{1}{c} (c\epsilon)^{2^m} = \exp(-L).$$

Include the residual $V$ into $H_{garbage}$. 
How to construct $U$ (quasi-adiabatic continuation):

For any fixed $H_0$ and $V$ the spectral gap $\Delta(\epsilon)$ of $H_0 + \epsilon V$ is a continuous function of $\epsilon$ such that $\Delta(0) \geq 1$.

We shall assume that the gap $\Delta(\lambda)$ is not too small on the interval $[0, \epsilon]$, say at least $1/3$, and use this assumption to show that that the gap $\Delta(\epsilon)$ is much larger than $1/3$, say, $\Delta(\epsilon) \geq 2/3$.

If this holds for all $\epsilon \in [0, \epsilon_0]$ then $\Delta(\epsilon) \notin [1/3, 2/3]$ for any $\epsilon \in [0, \epsilon_0]$. By continuity it implies $\Delta(\epsilon) \geq 2/3$ for all $\epsilon \in [0, \epsilon_0]$. 

![Graph showing the relationship between $\Delta$ and $\epsilon$](image)
How to construct $U$ (quasi-adiabatic continuation):

Now we are in the settings of the exact quasi-adiabatic evolution theorem:

$$\Psi(\epsilon) = U \cdot \Psi(0) \equiv U \cdot P,$$

where $U$ describes unitary evolution under (approximately) local Hamiltonian for time $O(1)$.

The Hamiltonian

$$\tilde{H} = U^\dagger (H_0 + \epsilon V) U$$

is globally block-diagonal, that is, $\tilde{H} \cdot P \subseteq P$.

This is almost what we need:

$$\tilde{H} = H_0 + W, \quad W = U^\dagger H_0 U - H_0 + \epsilon U^\dagger V U$$

$W$ has strength $O(\epsilon)$ and fast enough decay of interactions. However $W$ is only globally block-diagonal, $W \cdot P \subseteq P$. 
How to construct $U$ (quasi-adiabatic continuation):

Remaining step: reduction from global to local block-diagonality (the hard part)

$$\tilde{H} = \int_{-\infty}^{\infty} g(t) e^{i\tilde{H}t} \tilde{H} e^{-i\tilde{H}t}$$

Using the assumption that $\tilde{H}$ is gapped one can choose the filter function $g(t)$ such that

$$W = \sum_{u \in \Lambda} W_u, \quad W_u \cdot P \subseteq P, \quad W_u = \sum_{B \ni u} W_B,$$

The magnitude of $W_B$ decays fast enough for large clusters $B$. However individual terms $W_B$ do not preserve $P$.

One extra trick is needed to show that $W_u$ can be approximated by a locally block-diagonal operator. This approximation relies on TQO-1,2.
Conclusions

(1) The spectral gap of spin Hamiltonians composed of local commuting projectors and satisfying conditions TQO-1,2 does not close in a presence of generic local perturbations.

(2) Conditions TQO-1,2 can be extended to systems with symmetries.

(3) Lieb-Robinson bound and the quasi-adiabatic continuation permit analysis of perturbed quantum systems which does not rely on perturbative expansions.