Topological implications in quantum tomography

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Topological implications in quantum tomography

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Question: How many measurements/outcomes are necessary to identify a quantum state ρ under prior information $\rho \in M$?

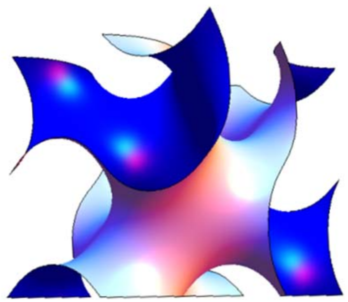
Setup:
- assume: prior info restricts to manifold M of dimensionality d_M
- measure **(i)** m expectation values or **(ii)** POVM with $m+1$ outcomes:

$$h : M \to \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho A_i]$$

Goal:
- find minimal m s.t. h is injective (info complete for M)

Example: M = pure states in $\mathbb{C}^d$: $d_M = 2d - 2 \leq m \leq d^2 - 1$

[Flammia et al.]: $2d - 1 \leq m$
[Gross et al.]: efficient probabilistic scheme with $m = O\big(d(\log d)^2\big)$



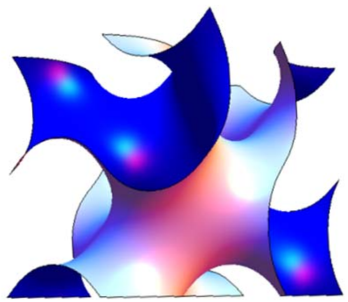
Topological obstructions

Proposition: $h : M \to \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho^{\otimes n} A_i]$
is info-complete for M iff it is a topological embedding

Recipe for lower bounds on m :

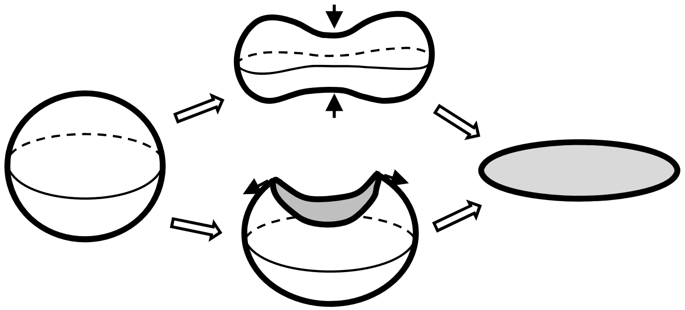
show that topological properties of M have no realization in too small dimensions m

Powerful toolboxes: homotopy, cohomology, etc.



Example 1: M = pure qubit states

$d_M = 2$



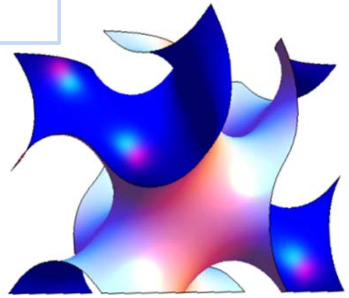
Observation: map from Bloch-SPHERE to $\mathbb{R}^2$ either discontinuous or not injective

i.e. $m > d_M$.

Corollary: $h: M \to \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho^{\otimes n} A_i]$

is info-complete for M iff it is so for **all** qubit states.

Borsuk-Ulam: If $m = 2$ then there exist two **orthogonal** states which cannot be distinguished.



Example 2: M = pure states in $\mathbb{C}^3$ with **real** amplitudes

$|\psi\rangle = (x, y, z) \in \mathbb{R}^3$

$d_M = 2$

Boy surface, Oberwolfach

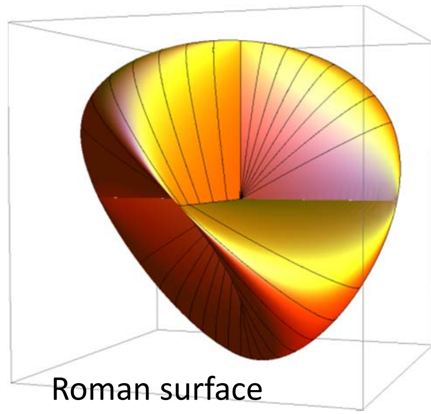
Observation: $M \simeq$ real projective plane $\mathbb{R}\mathbf{P}^2$

Corollary: $h : M \to \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho^{\otimes n} A_i]$

is info-complete for M only if $m \geq 4$. $m = 4$ can be realized for $n = 1$.

proof idea:

- non-orientability of $\mathbb{R}\mathbf{P}^2$ implies self-intersections in $\mathbb{R}^3$
- $(x, y, z) \mapsto (yz, xz, xy, x^2 - y^2)$ leads to **topological embedding**



Roman surface

Obstructions from differential topology

Proposition: With some assumptions on M, $h : M \to \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho A_i]$ is info-complete for M iff it is an embedding in the category of differential topology.

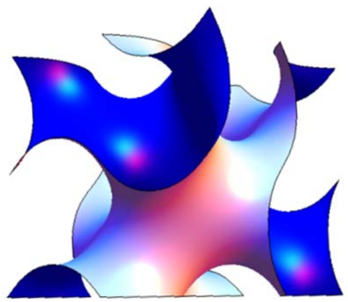
Assumptions:

- M is smooth submanifold
- Union of tangent spaces is contained in \`difference space´ $\{X | X = \lambda(M_1 - M_2),\ M_i \in M,\ \lambda > 0\}$

Lemma: This holds for $M = \mathbb{C}\mathbf{P}^{d-1}, G(r, d-r)$

Powerful toolboxes for lower bounds on m :

- Atiyah Hirzebruch index theorem
- Chern‘s results on dual Stiefel-Whitney classes



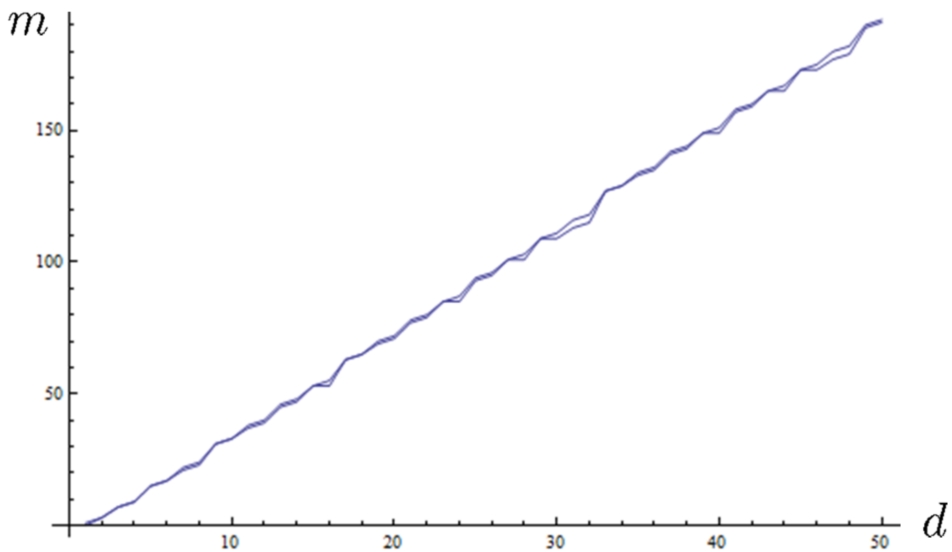
Pure states in $\mathbb{C}^d$

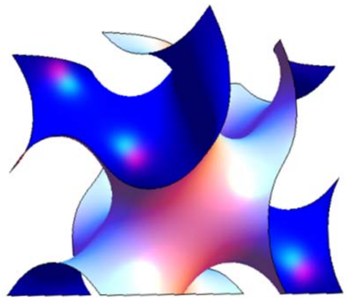
Proposition: The min m for which $h: M \rightarrow \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho A_i]$
can be info-complete satisfies

$$2d_M - 2\alpha < m \leq 2d_M - \alpha$$

where α = number of 1's in binary expansion of $d-1$

note: $\alpha \leq \log_2 d, \; d_M = 2d - 2$





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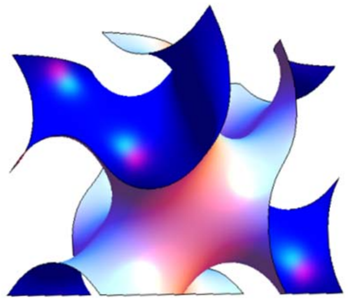
Remarks:

- Analogous result for states with rank constraint (via Grassmannians)
  In particular $m \leq 2d_M - 1, \; d_M = 2r(d-r)$
- Upper bounds via explicitly constructed observables



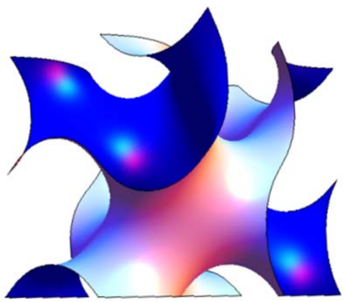
General upper bound

Let M be a set with Minkowski dimension

$D_M := \limsup_{\epsilon\to 0} \frac{\log N_\epsilon}{\log(1/\epsilon)}$, $N_\epsilon =$min number of covering ϵ balls

(note: $D_M = d_M$ for smooth manifolds)

Proposition: Almost every $h : M \to \mathbb{R}^m, \quad h(\rho)_i = \mathrm{tr}[\rho A_i]$
is info-complete for M if $m > 2D_M$



Conclusion

- Topological properties of *prior information* are relevant for min m

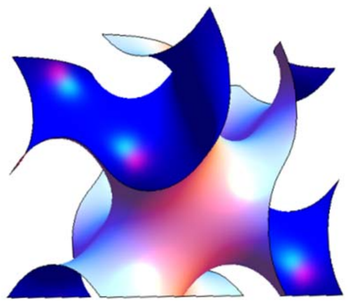
- m can exceed the number of parameters necessary for description by a factor of two but not more

- Results beat e.g. *compressed sensing*. However, we optimized m irrespective of classical post-processing, robustness and verifyability of assumptions

joint work of: Luca Mazzarella, Teiko Heinosaari, Michael Wolf

presentation: David Reeb

on arXiv soon …



Job announcement

where? TU Munich
what? postdocs & PhD's in QIT
when? from March on ...

contact: Michael Wolf (wolf.qit@gmail.com)

TUM