Localization assisted quantum error correction in the toric code

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Topological quantum error correcting codes are unique. Their non-trivial syndromes can be interpreted as quasiparticles, and logical errors correspond to the propagation of these in topologically non-trivial loops. Hence, to a much greater extent than other error correcting codes, a wealth of techniques from physics may be applied to understand the topological codes and enhance their power. Here we take such an approach, considering the effects of Anderson localization to suppress errors in the toric code. We find that without it, the quantum memory decoheres in a time linear with the system size when an external magnetic field is present. However, once the disorder inherent in physical realizations of the code is taken into account, the localization induced keeps the memory stable. The means by which the toric code may be redesigned in order to enhance this effect are then considered. Specifically, we consider the code designed on random lattices. It is found that this not only slows down the errors caused by an external field, but slows thermal errors also.

The toric code is a topological error correcting code capable of storing two logical qubits [1, 2]. It is defined on a two dimensional $L \times L$ lattice wrapped around a torus with a spin-1/2 on each edge. Stabilizers are defined on the spins around each plaquette and vertex. A non-trivial syndrome for any of these can be interpreted as the presence of a quasiparticle, known as e and m anyons on vertices and plaquettes, respectively. All properties of the code can then be understood in terms of these particles, with local errors causing them to be created in pairs and propagated around the lattice. Logical errors are formed when anyons move in topologically non-trivial paths, or when the density of anyons is so high that they cannot be reliable reannihilated without forming topologically non-trivial loops.

Since the stabilizers of the toric code are quasilocal, acting only on spins located around the same plaquette or vertex, they may be implemented in a Hamiltonian. This has the stabilizer space of the code as its ground state, and energetically penalise the creation of anyons. The energy gap will therefore suppress errors, and has been shown to be stable against local perturbations [3]. Even so, the dynamic effect of such perturbations on excited states of anyons is known not to enjoy the same stability [4]. Specifically we study here the effects of an external magnetic field on the anyons. We find that the field causes the anyons to move according to quantum walks [5]. This propagates them at a speed proportional to the strength of the field, and leads to logical errors in a time linear with the system size, L. Such a result is disastrous for the code, since only a single anyon pair needs to be created by a random process for the field to completely destroy the stored information. The critical density of anyons that the code can correct then becomes zero in the presence of a stray magnetic field. Realistically we must expect that, in any physical realization of the toric code, such stray fields will be present. It is therefore of vital importance to address this weakness.

Our approach is to apply the phenomenon of Anderson localization, a well known effect from condensed matter physics [6–11]. This predicts that the motion of quantum walks becomes exponentially suppressed in the presence of disorder. Specifically, the probability that a pair of anyons move a distance d from their initial positions in time t is bounded by,

$$P(d,t) \leq L^4 e^{-d/l}. \tag{1}$$

Here l is the so-called localization length of the system induced by the disorder. It is l that characterises the strength of the effect, with small values implying strong localization. Random fluctuations in the couplings of the toric code Hamiltonian induce a finite l, and can therefore be used to prevent anyons causing logical errors. This will allow the density of anyons to be tolerated by the code to remain finite, even in the presence of a field. However, the application of disorder must be done with care. Any attempts to purposefully induce disorder will have the side effect of reducing the gap, and so cause more harm than good. Only the disorder naturally present in any physical realization of the code may therefore be used. However, this can be expected to be weak, and so not induce strong enough localization to have a significant effect. We have therefore studied the means to make the most of this disorder, causing the anyons to be strongly localized.

In our study, we consider the perturbed toric code Hamiltonian to take the form,

$$H = -\sum_v J_v A_v - \sum_p J_p B_p + h \sum_i \sigma_i^z. \tag{2}$$

The choice of pertubation causes the quantum walks to be induced for any e anyons present on the system. Disorder in the J_v will therefore cause localization. We model this disorder by assigning each J_v a random value in the interval $J - \delta$ to $J + \delta$, according to a Cauchy distribution with width γ. Here J is the average value of the J_v, and the parameters δ and γ govern the strength of the disorder. For concreteness, the disorder is evaluated when $\gamma = \delta/10$ and $h = \delta/100$. This relations are sufficient to specify the exact effects of the disorder, and so specific values are not required. Also, the specific value of J is not required, except that $J > \delta$ must be satisfied to ensure that the Hamiltonian remains gapped. It is found that localization is indeed induced, and allows the code to tolerate a finite density of anyons. Specifically, error correction expected to succeed as while the density remains less than 10^{-3}. The fact that a finite density is tolerable in the presence of a field demonstrates the power of Anderson localization to successfully suppress errors.

The means to further strengthen the localization effect are then studied. We consider defining the toric code on random lattices, rather than the square lattice usually considered. On its own, this is found not to induce Anderson localization. However, it does significantly slow down the spread of errors, causing the time after which a single pair destroys the stored information to vary polynomially, rather than linearly, with L. As such it will enhance the localization effect induced by the disordered J_v.

Having demonstrated how disorder may be used to suppress the effects of coherent errors induced by a magnetic field, we turn our attention to incoherent errors, such as those caused by a constant temperature. Specifically we model the effects of thermal errors on the toric code, which move the anyons according to classical random walks. This is done for both the square and random lattices. Though, since the walks are not coherent, an effect such as Anderson localization does not occur. However, the random lattices are numerically seen to suppress logical errors. We find that the critical time after which the thermal errors are uncorrectable is larger for the random lattice than the square lattice. This opens up an interesting topic of research, to see how disorder in lattices affects classical walks of anyons, and whether it may lead to a new approach for thermally stable quantum memories [12]. If so, this will compliment existing approaches [13, 14], and bring the realization of quantum computation ever closer.

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