Quantum Strategic Game Theory

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Background

Game theory. Game theory is a branch of applied mathematics to model and analyze interactions of two or more individuals, usually called *players*, each with a possibly different goal. Over decades of development, game theory has grown into a rich field on its own, and has found numerous applications in economics, political science, biology, philosophy, statistics, computer science, etc. Many models have been proposed to study games, among which the most popular and fundamental ones are *strategic games* (or games in strategic or normal form) and *extensive games* (or games in extensive form). In the former the players choose their strategies simultaneously, and then each receives a payoff based on all players' strategies. In the latter the players choose their strategies adaptively in turn, and finally when all players finish their moves, each receives a payoff based on the entire history of strategies of all players. Various settings are also studied. For instance, if before playing the game, each player also receives a private and random input, then they are playing a *Bayesian game*, which belongs to the larger class of *games with incomplete information.*

Equilibrium as a central solution concept in game theory attempts to capture the situation that each player has adopted a strategy that is optimal with respect to their own goals. Nash equilibrium (NE) is the first and most fundamental concept of equilibrium, and it has many extensions such as correlated equilibrium (CE) which relaxes the requirement of the probability distribution being independent among the players.

Partially motivated by the emergence of Internet and other systems with a huge number of players, the concepts of complexity in computer science has been added as one more dimension for studying games. Many central concepts such as Nash equilibria and important areas such as mechanism design has been revisited with the algorithmic ingredient injected.

Quantum games: There are mainly two tracks of literature under this name.

1. *Nonlocal games*, a particular class of Bayesian games in the strategic form. In these games, each of the two or more parties receives a private input drawn from some known distribution, and the players output random variables, targeting a particular correlation between their outputs and inputs. The main goal of designing and studying these games is to show that some correlations are achievable by quantum entanglement but not classical randomness, thus providing more examples for Bell's theorem that refutes Einstein's program of modeling quantum mechanics as a classical theory with hidden variables.

2. *Quantization of strategic games.* Unlike the first track of research motivated by physics (and computational complexity theory), the second track of work aims at quantizing classical strategic game theory. There are various models proposed, but the basic approach is to allow each player i to take quantum operations on her own space $H_i = span(S_i)$, where S_i is her set of classical strategies. Then a measurement is made to get a (random) classical joint strategy s, which decides the payoff of the players by the classical payoff functions u_i. Variation exists in details of different proposed models. For example, in the seminal paper by Eisert, Wilkens and Lewenstein and many follow-ups, there is an extra party, called Referee, who generates a state $J|0\rangle$ by some unitary

operation J in the Hilbert space of dimension $\sum_i |S_i|$, and partitions the state into k parts (of the dimensions $|S_1|, \ldots, |S_k|$ respectively) for the k players. After the k players perform their individual quantum operations, Referee collects the parts and performs the inverse operation $J^\dagger$ on the state before measuring on the computational basis. The work also assumes that each player only take a particular subset of unitary operations.

There are many follow-up work on the same or similar models. Despite the rapid accumulation of literature, controversy also exist. For example, Benjamin and Hayden found that if the assumption on available strategies in is removed, then the proposed state is not an NE any more.

There are actually more literature related to quantum games, which are sporadic on topics, and has a fewer number of papers on each fixed topic. We just give two examples here. The first one is the widely-referenced seminal work by Meyer, who demonstrates the power of using quantum strategies under some particular restriction on the other player's strategies as well as the sequence of moves. Another work, which has a closer relation to computer science, studies the general refereed game in the extensive form. The model adopted there is very general, easily encompassing all previous work (and the model in our paper) as special cases. It has interesting applications such as a very short and elegant proof of Kitaev's lower bound for strong coin-flipping. Of course, the generality also makes it very challenging to discover many more nontrivial properties.

Our Results

To address the above concerns, we study the quantum strategic games with emphasis on the asymptotic properties of general games with the least assumption about the quantum strategies. Recall that in a classical game, Player i chooses a strategy from a set S_i, and the joint strategy decides Player i's payoff by the payoff function $u_i(s)$. Now in our quantum model, starting from a joint state ρ, each player i can apply a general CPTP map in her local space $H_i = span(S_i)$, and then a measurement is made on the computational basis, getting a random joint strategy s. Player i then gets an expected payoff $\mathbf{E}_s[u_i(s)]$. Different than the EWL-model which only allows a proper subset of unitary operations, we allow arbitrary CPTP maps for each player. In addition, no referee is involved; namely the operations J and $\hat{J}$ are removed. We find this corresponds to the classical model more naturally because, after all, a classical game does not have a classical referee to shuffle the whole strategy space $S = \times S_i$ before and after the players choose their strategies.

Apart from the model, what mainly distinguishes the present work from the previous ones is the generality of the *classical* games under quantization. Most of the previous work focus on particular games, usually of small and fixed sizes. While it is surely natural to start at particular and small examples, it is clearly desirable to have a uniform treatment and to understand general properties, especially when the game has large set of strategies. A fundamental question is:

Question: How much "advantage" can playing quantum strategies provide, if any?

Depending on how the advantage is measured, we study the question in two ways.

1. The advantage is measured by the increase of payoff. We defined the notion of strategic quantum equilibria as follows. A quantum state ρ in space $H = \otimes_i H_i$ is a quantum correlated equilibrium if Player i cannot increase her expected payoff by any local operation. If further $\rho = \otimes_i \rho_i$ for some ρ_i in H_i, then it is a quantum Nash equilibrium. Under this definition, we studied various relations between classical and quantum equilibria in both qualitative and quantitative ways. Given a quantum state, the most natural classical distribution it induces is given by the measurement on the computational basis S. That is, ρ induces p defined by $p(s) = \rho_{s,s}$. Not surprisingly, one can show that if ρ is a quantum Nash (or correlated) equilibrium then p is a classical Nash (or correlated) equilibrium.

The other direction, namely transition from classical to quantum, is more complicated but more interesting. A classical distribution p over S have two natural quantum counterparts: 1) *classical mixture*: $\rho(p) = \sum_s p(s)|s\rangle\langle s|$, the mixture of the classical states, and 2) *quantum superposition*: $|\psi(p)\rangle = \sum_s \sqrt{p(s)}|s\rangle$, most commonly used superposition of the classical states. We regard the second mapping as more important because first, this is really quantum, and second this mapping is the most commonly used quantum superposition of a classical distribution in known quantum algorithms. It so happens that it is also the most intriguing case of our later theorems. 3) One can also consider the broad class of quantum states ρ satisfying $p(s) = \rho_{s,s}$, including the above two concrete functions as special cases. Now the question is, do the transformations keep the Nash/correlated equilibrium properties? It turns out that the classical mixture mapping keeps both Nash and correlated equilibria, but the quantum superposition mapping only keeps Nash equilibria. As to the general class of correspondence, no equilibrium is guaranteed to be kept.

The above relations between classical and quantum equilibria are qualitative aspect, but it is natural to care about them also quantitatively: After all, if $|\psi(p)\rangle$ is not a quantum exact correlated equilibrium but an ϵ-approximate one for some small ϵ, then the interest of using quantum strategies significantly drops since the quantum advantage is small. In this regard, we ask the following natural question: In a [0,1]-normalized game, what is the largest gain of payoff by playing a quantum strategy on a state corresponding to a classical equilibrium?

Theorem 0.1 *In a $[0,1]$-normalized $(n\times n)$-bimatrix game, 1. there exists a CE p s.t. $u_1(|\psi(p)\rangle) = \tilde{O}(1/\log n)$ but $u_1(\Phi_1(|\psi(p)\rangle)) = 1 - \tilde{O}(1/\log n)$ for some local operation Φ_1, and a CE p s.t. $u_1(\Phi_1(|\psi(p')\rangle))/u_1(|\psi(p')\rangle) = n^{0.585\ldots}$.*

2. There exists a Nash equilibrium p, and a quantum state ρ with $\rho_{ss} = p(s)$, s.t. $u_1(\rho) = 1/n$ but $u_1(\Phi_1(\rho)) = 1$ for some local quantum operation Φ_1. The additive increase of $1 - 1/n$ and the multiplicative increase of n are the largest possible even for all correlated equilibria p.

The generality of the game, though desirable, also gives us challenges: Finding the maximum quantum advantage requires to solve a non-convex programming, which is infeasible in general. The main approach for Part 1 is to construct a large game from smaller ones. Various constructions exist; what we need is to preserve the equilibrium *and* increase the gap. It turns out that tensor product satisfies both criteria, thus the task reduces to study a smaller size problem, with the aim not at optimizing this small game but optimizing the large one constructed from this small game. So it becomes designing and analyzing games with original payoff very close to 1 and incentive very close to 0, which, when taken power, gives large incentive.

2. The advantage is measured by the hardness of generating a correlated equilibrium. We propose a new complexity measure, *correlation complexity*, and show that some correlated equilibria is hard to generate classically but easy to generate quantum mechanically. For a distribution $p = (X, Y)$, its correlation complexity is defined as the minimum size of "seed correlation" (X', Y') *s.t.* local operations on X' and Y' are able to output a sample of (X, Y). Note that the key difference to nonlocal games is that no *private* input is given to Alice and Bob, thus the model is simpler and thus arguably more fundamental. It turns out that even such a basic model can still be used to demonstrate the power of quantum mechanics by admitting a large separation between randomized and quantum correlation complexities.

We give bounds for the quantum correlation complexity by matrix rank, and fully characterize the randomized correlation complexity by nonnegative rank, a well-studied measure in linear algebra with many applications to many other fields. Then we exhibit a family of n-bit distributions which can be generated by one EPR pair, but at least $\log_2 n$ classical shared random bits. We conjecture that a random distribution with quantum correlation complexity 1 has randomized correlation complexity exactly n.