

Isotropic Entanglement

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We propose a method which we call “**Isotropic Entanglement**” (IE), that takes inspiration from Free Probability Theory and other ideas in Random Matrix Theory, in order to predict the eigenvalue distribution of quantum many body systems (QMBS) with generic interactions. At the heart of this method is a sliding scale (“the Slider”) which matches fourth moments. We provide examples that show IE provides an accurate picture of the spectra well beyond what one expects from the first four moments alone.

QMBS spectra have been elusive for two reasons: 1. Standard numerical diagonalization is intractable, for even a moderate number of particles, because of exponential growth. 2. The terms that represent the interactions are generally non-commuting. This is particularly pronounced for systems with random interactions (e.g., Quantum Spin Glasses). Indeed, the interaction is the very source of quantum entanglement, which makes the study of strongly correlated system in condensed matter physics a formidable task.

The applicability may not be restricted to one dimensional chains, but for the sake of concreteness, we investigate N interacting d -dimensional quantum spins (qudits) on a 1-dimensional lattice with generic nearest neighbors interaction. Mathematically the Hamiltonian can be written

$$H = \sum_{l=1}^{N-1} \mathbb{I}_{d^{l-1}} \otimes H_{l,l+1} \otimes \mathbb{I}_{d^{N-l-1}}, \quad (1)$$

where the local terms $H_{l,l+1}$ are known random matrices.

A naive way of approximating the eigenvalues of H is to treat it *classically*: form all possible sums of eigenvalues from each summand. This gives the exact answer only when the summands in (1) commute. The eigenvalue distribution of any commuting subset of H such as odds (or even) can be obtained *classically* by forming a Cartesian sum of the eigenvalues of the summands giving the random matrices A (or B) or equivalently taking a classical convolution of the densities of the local terms. However, the difficulty in approximating the full spectrum of H is in summing the odds and the evens because of their overlap at every site.

The intuition behind this method is that terms with overlap, such as $H_{l,l+1}$ and $H_{l+1,l+2}$, introduce randomness and mixing through sharing of the sites. In other words, the process of entanglement generation introduces an *isotropicity* between the eigenvectors of evens and odds. The random distribution of the local terms and the overlaps of even and odd terms at every site introduce a sort of *genericity* that we wanted to harness.

One has several options in approximating the distribution of the sum (i.e., the spectrum of H). One extreme is to add them classically ($p = 1$), by which one fully ignores the non-commutativity of neighboring terms. This implies that A and B can be diagonalized by the same set of eigenvectors. The other extreme is to add them *isotropically* ($p = 0$), by which one assumes that the matrices A and B are isotropic (or *free* in the case of infinite matrices). In this case the eigenvectors of A and B are in generic positions with respect to one another,

$$H^c = A + B \quad H^{\text{iso}} = A + Q^T B Q \quad H = A + Q_q^T B Q_q$$

where, Q is a Haar measure orthogonal matrix of size d^N , $Q_q = \left(Q_q^{(A)}\right)^T Q_q^{(B)}$, where due to their overlap $Q_q^{(A)}$ and $Q_q^{(B)}$ cause entanglement in the chain and A and B are assumed with no loss of generality to be diagonal. Mathematically

$$Q_q^{(A)} = \left[\otimes_{i=1}^{(N-1)/2} Q_i^{(O)}\right] \otimes \mathbb{I}_d \text{ and } Q_q^{(B)} = \mathbb{I}_d \otimes \left[\otimes_{i=1}^{(N-1)/2} Q_i^{(E)}\right] \quad N \text{ odd}$$

$$Q_q^{(A)} = \left[\otimes_{i=1}^{N/2} Q_i^{(O)}\right] \text{ and } Q_q^{(B)} = \mathbb{I}_d \otimes \left[\otimes_{i=1}^{N/2-1} Q_i^{(E)}\right] \otimes \mathbb{I}_d \quad N \text{ even,}$$

where each $Q_i^{(\bullet)}$ is a Haar measure orthogonal matrix of size d^2 and \mathbb{I}_d is an identity matrix of size d . The tensor product in the last equations succinctly summarizes the departure of the quantum case from a generic matrix as well as from the classical case. First of all the number of parameters in Q_q grows linearly with N whereas it grows exponentially with N for Q . Second, the quantum case yields isotropicity that makes it very different from the classical case, whose eigenvectors represent a point on the orthogonal group (i.e., the identity matrix). Lastly we can replace Haar measure on the orthogonals with Haar measure on the unitaries so that the complex case is covered by the same theory.

Neither of the two extremes give a satisfactory approximation of the spectrum of the actual problem (Eq. 1). However, a convex combination of the two gives a very good approximation. Existing tools of Random Matrix Theory (RMT) often rely on eigenvectors with Haar measure, the uniform measure on the orthogonal/unitary group. However, the eigenvectors of QMBS have a more special structure that is characteristically different from RMT and the classical treatment. Therefore we created a *hybrid theory*, that is in between, where the entanglement generating summands in (1) are treated, say, with a finite version of Free Probability Theory (FPT). As a prelude see Figure 1 that compares the IE method with exact diagonalization. The *Slider* displays the proposed mixture.

We will show that the *classical*, *isotropic* and the desired *quantum* problems depart in their fourth moments (The Departure Theorem). We then calculate the fourth moments of the three cases and prove that the quantum problem lies in between the two (The Slider Theorem.) The

method of isotropic entanglement provides the parameter $0 \leq p \leq 1$, which indicates the mixture of classical and isotropic treatment by matching the fourth moments. Luckily and interestingly the kurtosis of the quantum case lies in between the classical and the free. Therefore, we want to use the knowledge of the kurtosis of the quantum case (γ_2^q) in terms of the kurtoses of the classical (γ_2^c) and the isotropic (γ_2^{iso}) to form the correct hybrid theory, whereby we need to optimize the parameter $0 \leq p \leq 1$ such that

$$\gamma_2^q = p\gamma_2^c + (1 - p)\gamma_2^{iso} \Rightarrow p = \frac{\gamma_2^q - \gamma_2^{iso}}{\gamma_2^c - \gamma_2^{iso}}. \quad (2)$$

This can be evaluated from the fourth moments:

$$m_4^{(c)} = \frac{1}{d^N} \mathbb{E} \text{Tr} (A + B)^4, \quad m_4^{(iso)} = \frac{1}{d^N} \mathbb{E} \text{Tr} (A + Q^T B Q)^4, \quad m_4^{(q)} = \frac{1}{d^N} \mathbb{E} \text{Tr} (A + Q_q^T B Q_q)^4 \quad (3)$$

The conclusion arrived at here is another example of the principle that a small subset of the underlying space suffices in capturing the essentials for most problems encountered in studying QMBS. This realization, arrived at via a different route, is at the heart of the recent developments in QMBS research such as Matrix Product States, Density Matrix Renormalization Group, where the *state* (usually the ground state of 1D chains) can be adequately represented by a Matrix Product State (MPS) ansatz whose parameters grow linearly with the number of quantum particles.

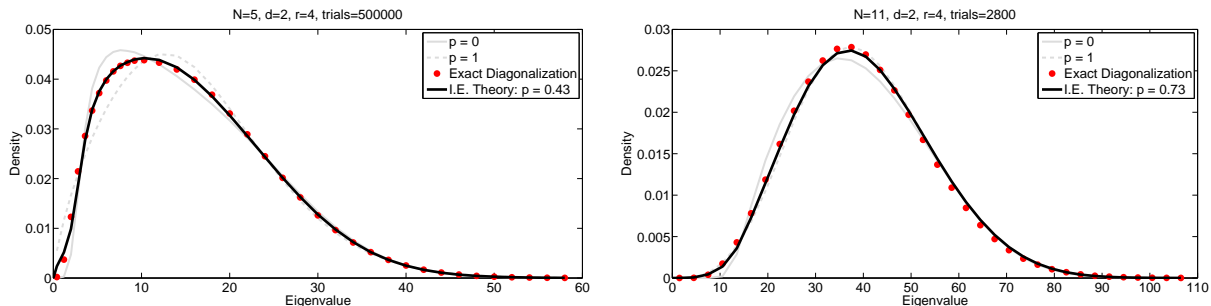


Fig. 1.— *Prelude*: The normalized histogram of the eigenvalues for two chains with lengths N and Wishart matrices of different ranks r as local terms. I.E. makes a convex combination of the isotropic ($p = 0$) and classical ($p = 1$) to capture the quantum spectrum.