Quantum query complexity of minor-closed graph properties

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Graphs

In this talk, graphs will be undirected and simple.

(i.e., no self loops or multiple edges between vertices)

A graph on n vertices can be specified by n(n-1)/2 bits.

A graph property: A (nontrivial) map from the set of all graphs to {0,1} that maps isomorphic graphs to the same value (i.e., a property that is independent of labeling).

Examples of graph properties: Planarity, bipartiteness, k-colorability, connectivity, etc.

Non-examples: "the first vertex is isolated", "odd-numbered vertices have even degree"

Query complexity of graph properties

The query complexity model: We can query a black box with a pair of vertices (i,j) to find out if there is an edge between them.

All graph properties can be decided with n(n-1)/2 queries.

D(P), R(P), Q(P): Deterministic, randomized and quantum query complexities of determining property P

$Q(P) \leq R(P) \leq D(P) \leq n(n-1)/2 = O(n^2)$

Example: If P is the property of being the empty graph (i.e., the property of not containing any edges), then

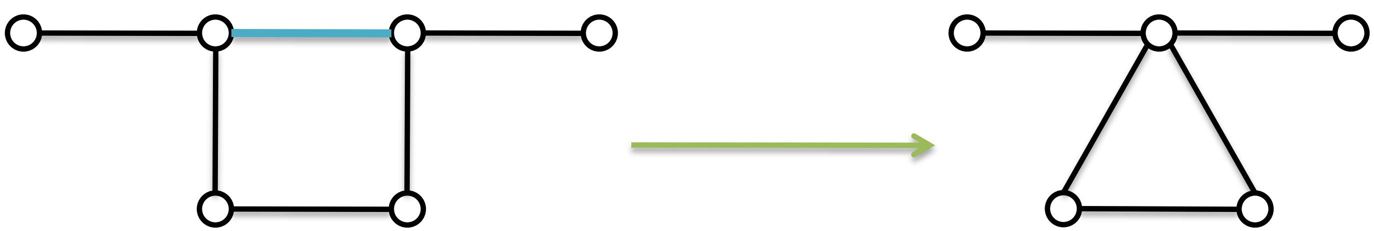
$D(P) = n(n-1)/2$, $R(P) = \Theta(n^2)$ and $Q(P) = \Theta(n)$

Graph minors

Subgraph: A graph that can be obtained by deleting edges and deleting isolated vertices.

Minor: A graph that can be obtained by deleting edges, deleting isolated vertices and contracting edges.

Edge contraction:



Minor-closed property: All minors of a graph possessing such a property also possess the property

Examples: Planarity, acyclicity (property of being a forest), property of being embeddable on a torus, etc.

Forbidden minors & forbidden subgraphs

Planarity is characterized by forbidden minors: G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

G is a forest if and only if it does not contain C_3 as a minor.

Robertson-Seymour theorem [1983-2004, ≈ 500 pages]:

All minor-closed properties are characterized by a finite set of forbidden minors.

Forbidden subgraph property (FSP): A property that can be characterized by a finite set of forbidden subgraphs

Some properties are both minor closed and FSP, e.g.:

- The property of being the empty graph
- The property of having max degree ≤ 1

Subgraph-closed and sparse properties

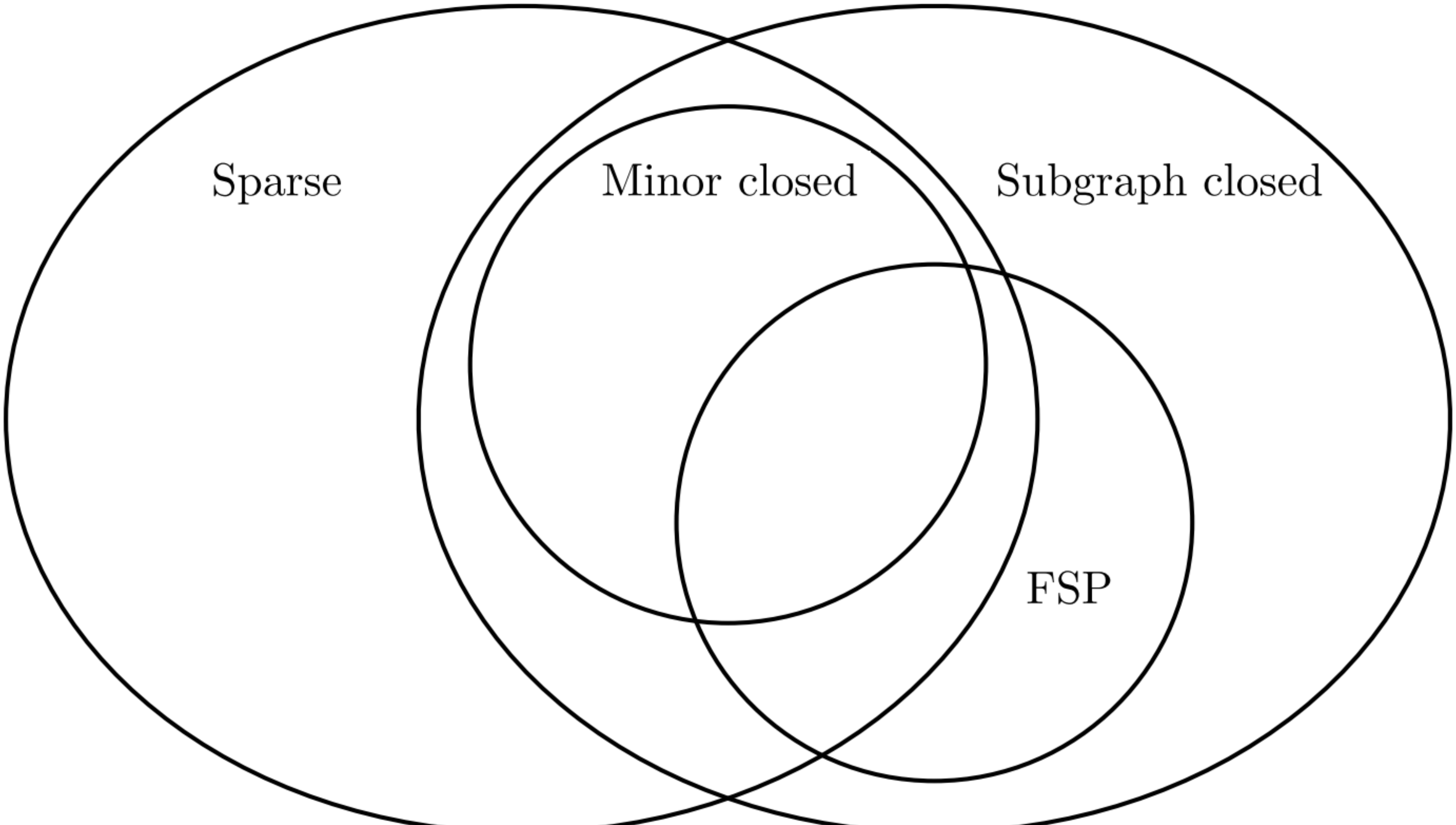
Subgraph-closed properties are closed under the subgraph operation, i.e., if G possesses such a property, then all subgraphs of G also possess it.

Examples: All minor-closed properties, all FSPs, bipartiteness

Sparse property: A property that can only be possessed by sparse graphs, where "sparse" means $|E| = O(|V|)$

Examples: Planarity (planar graphs have $|E| \leq 3|V| - 6$), Emptiness ($|E| = 0$), all minor-closed properties (by Mader's theorem), k-regular graphs for any fixed k ($|E|=k|V|/2$)

Venn diagram of graph properties



Simple observations

For sparse graph properties P, $Q(P) = O(n^{3/2})$

Since there are $n(n-1)/2$ potential edges, and $O(n)$ edges, this is the problem of finding K marked items in a list of size N, which requires sqrt(NK) queries, which is $O(n^{3/2})$ queries.

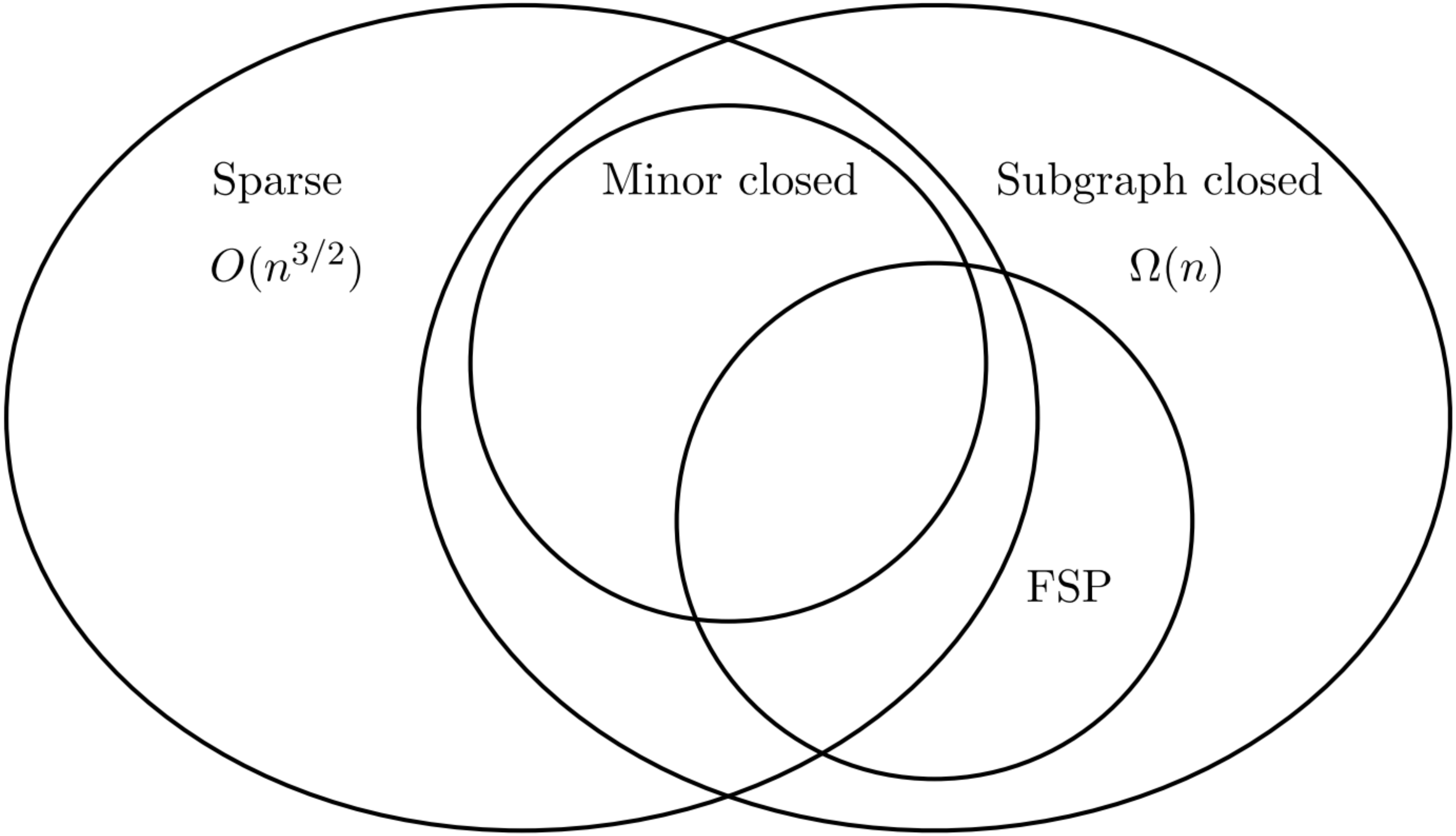
$\Rightarrow$ Minor-closed properties need at most $O(n^{3/2})$ queries

Subgraph-closed properties require $\Omega(n)$ queries

Proof idea: Since the property is subgraph closed, the empty graph possesses the property. Since this is a nontrivial property, there is a graph that does not possess this property. Distinguishing these two is hard (somewhat like the search problem).

$\Rightarrow$ Minor-closed properties require $\Omega(n)$ queries

Venn diagram of graph properties



For all minor-closed properties P: $Q(P)=\Omega(n)$, $Q(P)=O(n^{3/2})$

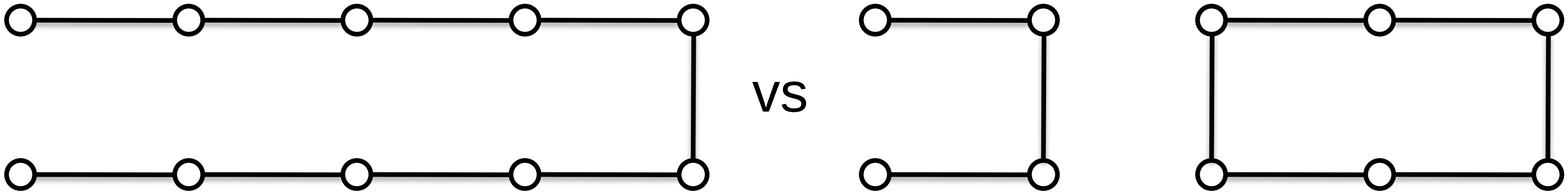
Query complexity of ACYCLICITY

The quantum query complexity of ACYCLICITY is $\Theta(n^{3/2})$.

Proof idea: Use the adversary method. Use a hard-to-distinguish set of cyclic graphs and set of acyclic graphs.

Distinguishing a long path from a long cycle is hard given only local information. However, paths contain degree-1 vertices which can be detected in O(n) queries.

Instead, use a path and a disjoint union of a path and cycle. (This proof is similar to the lower bound in Dürr et al. 2006)



Some minor-closed properties

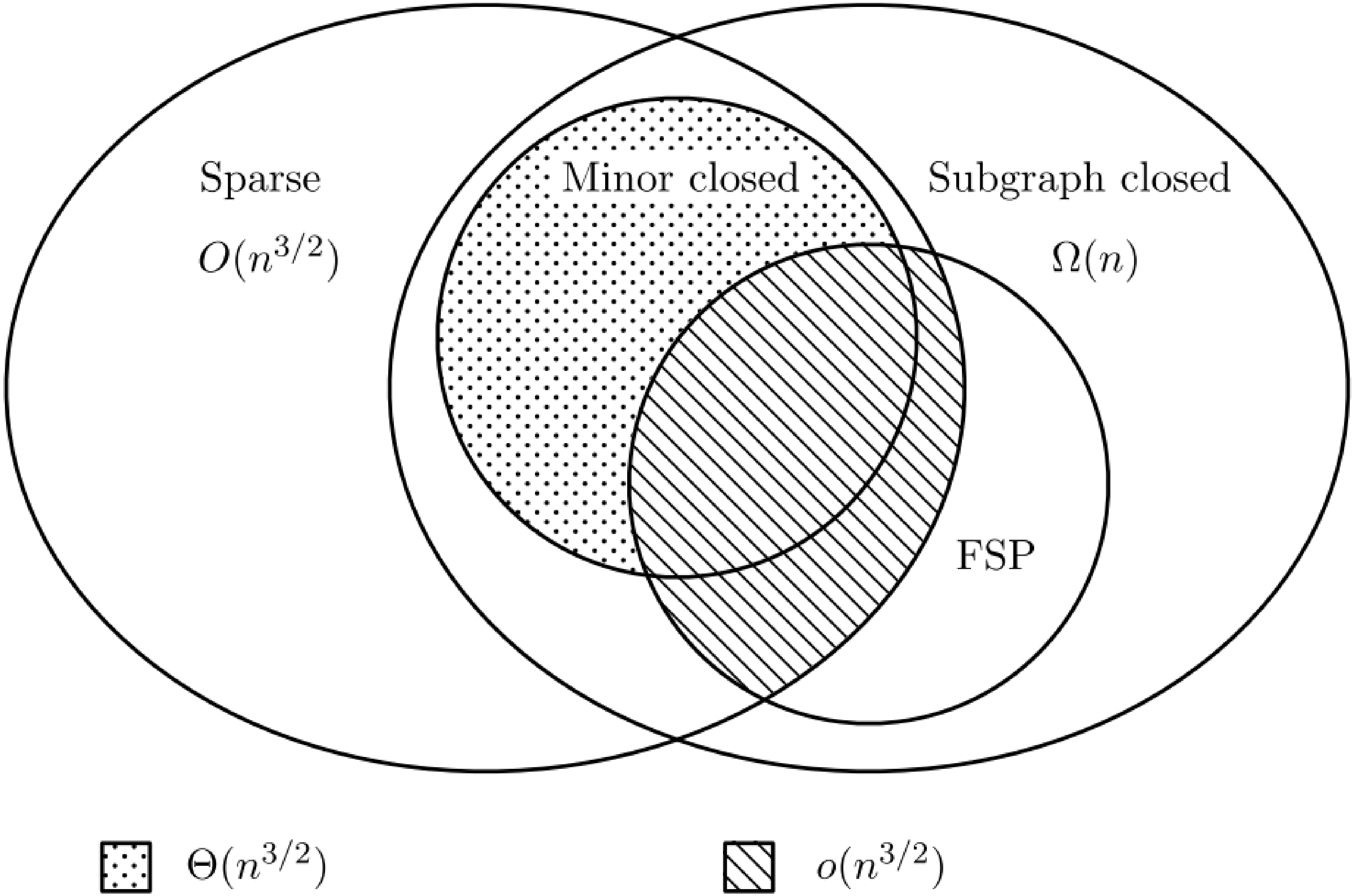
- Q(PLANARITY) = $\Theta(n^{3/2})$ [Ambainis et al. 2008]
- Q(ACYCLICITY) = $\Theta(n^{3/2})$ [Previous slide]
- Q(EMPTINESS) = $\Theta(n)$ [Same as the search problem]
- Q(MaxDegree ≤ 1) = $\Theta(n)$ [Search for a vertex of degree 2]

Observation: The first two properties are minor closed but not FSP, while the next two are both.

Theorem:
~~Conjecture:~~ Minor-closed properties that are not FSP have quantum query complexity $\Theta(n^{3/2})$.

Furthermore, minor-closed properties that are FSP can be recognized with $o(n^{3/2})$ queries. (More generally, this holds for sparse FSPs.)

Venn diagram of graph properties



Quantum walk for sparse FSPs

Consider the FSP property "does not contain H as a subgraph", where H is a fixed graph on (say) 5 vertices.

- Let the vertices of H be v_1, v_2, v_3, v_4 and v_5.
- We want to check if the given graph G contains H.

Assume G contains H and try to find it.

- Let the vertices of G that are isomorphic to v_1, v_2, v_3, v_4 and v_5 in H be u_1, u_2, u_3, u_4 and u_5 respectively.
- Assume we know the approximate (up to a multiplicative factor of 2) degrees of u_1, u_2, u_3, u_4 and u_5. Let these degrees be q_1, q_2, q_3, q_4 and q_5.
- Let the number of vertices of degree q_i in G be t_i.

Quantum walk for sparse FSPs

Our quantum walk algorithm:

- Set up 5 different quantum walks, each searching for one of the 5 vertices of H.
- Each walk searches over all t_i vertices of degree q_i for the vertex u_i by storing subsets of vertices.
- Every few steps, the different walks talk to each other and check if they have found 5 compatible vertices.

Some salient features of the walks:

- The 5 walks proceed at different speeds (depending on the values of t_i and q_i).
- Sparsity of G is essential: Searching for high-degree vertices is expensive, but such vertices are rare.

Other applications of our framework

C_4 finding problem: Natural generalization of C_3 finding.

$Q(C_3 \text{ finding}) = O(n^{1.3})$ [Magniez–Santha–Szegedy 2007]

$Q(C_4 \text{ finding}) = O(n^{1.25})$

Searching for bipartite graphs: Does the input graph contain a bipartite graph H as a subgraph?

The framework provides the best known algorithm for this problem.

Key idea: The framework applies because graphs excluding a bipartite graph cannot be too dense due to an extremal graph theory result (Kövári-Sós-Turán theorem).

Open problems

1. What is the complexity of minor-closed properties that are FSP? Somewhere between $\Omega(n)$ and $o(n^{3/2})$. Can all such properties be recognized in $O(n)$ queries?
2. Some specific examples like "does the graph contain a path of length k" may be easier to handle. Can we improve the upper bounds for these properties?
3. Can we improve the $\Omega(n)$ lower bound for <u>any</u> FSP? No such lower bound is known. (This cannot be done with the positive weights adversary method due to the certificate complexity barrier.)
4. The triangle-finding problem: Does the graph contain C_3 as a subgraph? Known bounds: $\Omega(n)$, $O(n^{1.3})$.

Thank you