The 2D AKLT state is a universal quantum computational resource

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I. MOTIVATIONS

Quantum computation promises exponential speedup over classical computation by exploiting the quantum mechanical nature of physical processes [1]. In addition to the standard circuit models, surprisingly, local measurement alone provides the same power of computation, given only a prior sufficiently entangled state [2]. Cluster states are the first known resource states for such measurement-based quantum computation (MBQC) [2, 3]. They can arise as the unique ground state of *five-body* interacting Hamiltonian on a square lattice; however, they cannot be the exact unique ground state of any *two-body* Hamiltonian [4, 5]. Fundamentally, could there be unique ground states of any two-body interacting Hamiltonian that are universal resources?

In searching for such resourceful ground states of physically reasonable Hamiltonians, Chen et al. made some important progress by constructing a spin-5/2 resourceful state on a honeycomb lattice, which is an unique ground state of a two-body interacting Hamiltonian [6]. Later Cai et al. approached this issue by patching ground states of Affleck-Kennedy-Lieb-Tasaki (AKLT) chains [7] into an effective 2D spin-3/2 state [8]. This construction reduced the local Hilbert-space dimension from 6 of Chen et al. to 4. However, both engineered Hamiltonians, even though consisting of only two-body interaction, turn out be complicated and possess less symmetry than the original AKLT Hamiltonians.

After many works on utilizing AKLT chains for quantum computation [9–11], it remains open whether any of the original 2D AKLT states can be universal resources for MBQC.

II. RESULTS

Our main result is that the ground state of the AKLT model (of spin-3/2) on the 2D honeycomb lattice can be reduced to a two-dimensional cluster state by local operations, and therefore is a universal resource state for MBQC. This transformation proceeds in two steps. (i) First, the AKLT state is mapped to a random encoded graph state by a local positive-operator-value-measure (POVM) measurement. (ii) Second, if the associated graph is sufficiently connected, the graph state can be further transformed by local measurements into a two-dimensional cluster state. In a Monte Carlo simulation, we demonstrated that the required connectivity properties hold for typical graphs. Beyond honeycomb, our method applies to any trivalent lattice, such as Archimedean lattices: $(3, 12^2)$, $(4, 6, 12)$ and $(4, 8^2)$, having bond percolation thresholds greater than 2/3 [12].

The significance of our results is that the ground state for the 2D AKLT spin-3/2 Hamiltonian $\sum_{(i,j)\in E(\mathcal{L})} f(\vec{S}_i\cdot\vec{S}_j)$, where $f(x)$ is a third-order polynomial [7], can be used as a quantum computational resource. The Hamiltonian is highly symmetric, i.e., invariant under rotation and is only nearest-neighbor interacting. The ground state is known to be unique for periodic boundary conditions [7]. The local Hilbert space dimension for such two-body interacting ground-state resource is also among the lowest that have been found so far, i.e., 4 [8]. Ours is probably the first one that shows a physically motivated quantum state in condensed-matter physics and in higher than one dimension turns out to provide a universal resource for measurement-based quantum computation.

III. INTUITIVE EXPLANATIONS

In this section we give slightly more detailed description and intuitive explanations for our results.

A. AKLT states

The AKLT state on the two-dimensional honeycomb $\mathcal{L}$ can be described in the following way. First, each vertex or site v of $\mathcal{L}$ contains three virtual spin-1/2 particles, lying at the ends of the three incoming edges (or bonds); see Fig. 1. The two virtual spins residing on the two ends of an edge $e = \{u, v\} \in E(\mathcal{L})$ linking the two nearest neighbors u and v are in the singlet state: $|\phi\rangle_e \equiv |01\rangle - |10\rangle$ (omitting the normalization). (Note that $|0\rangle \equiv |\uparrow\rangle$ and $|1\rangle \equiv |\downarrow\rangle$ are the two basis states of spin-1/2.)

Then at each lattice site v, a projection is made on the three virtual spins into the symmetric subspace

$$\Pi_{S,v} \equiv |000\rangle\langle 000| + |W\rangle\langle W| + |\overline{W}\rangle\langle\overline{W}| + |111\rangle\langle 111|, \quad (1)$$

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \quad (2)$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle), \quad (3)$$

where the four states $|000\rangle$, $|111\rangle$, $|W\rangle$ and $|\overline{W}\rangle$ constitute the basis states for the symmetric subspace of three spin-1/2 particles and they can also be regarded as the four basis states for a spin-3/2 particle $|3/2, 3/2\rangle$, $|3/2, -3/2\rangle$, $|3/2, 1/2\rangle$ and $|3/2, -1/2\rangle$, respectively, via the standard addition of angular momenta, where we have assumed implicitly the quantization is along z-axis.

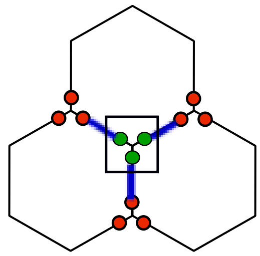

FIG. 1: Schematic picture of AKLT state on the honeycomb lattice $\mathcal{L}$. Each site contains three virtual spin-1/2 particles, each of which forms a singlet bond $|01\rangle - |10\rangle$ with the neighboring virtual spin-1/2. The rectangle represents a projection from the three virtual spins to their symmetric subspace, resulting in a local four-level (spin-3/2) system. Only a region on the honeycomb lattice is shown. The boundary condition can be chosen to be periodic or open and terminated by spin-1/2 particles.

To map the three virtual spin-1/2 particles into a physical spin-3/2 particle, only a relabeling is needed:

$$P_v = |3/2\rangle\langle 000| + |-3/2\rangle\langle 111| + |1/2\rangle\langle W| + |-1/2\rangle\langle \overline{W}|. \quad (4)$$

Thus the AKLT state on the honeycomb lattice can be viewed as one with a spin-3/2 per site (which can be regarded as composed of three virtual spin-1/2 particles), written as

$$|\mathrm{AKLT}\rangle := \bigotimes_{v\in V(\mathcal{L})} P_v \Pi_{S,v} \bigotimes_{e\in E(\mathcal{L})} |\phi\rangle_e, \quad (5)$$

where we use $V(\mathcal{L})$ and $E(\mathcal{L})$ to denote the set of vertices and edges, respectively, of graph $\mathcal{L}$.

B. POVM

In order to convert the state of multiple 4-level (spin-3/2) particles to that of qubits, we need to preserve a local two-dimensional structure. Naively speaking, this can be achieved by rank-2 projectors, such as $|3/2\rangle\langle 3/2| + |-3/2\rangle\langle -3/2|$. However, there is a finite probability that the state fails to be projected in the desired subspace. This motivates us to employ a generalized measurement or POVM to maximize the probability of useful outcome. In fact the following POVM, $\mathbb{1} = E_x + E_y + E_z$ with $E_\mu = F_\mu^\dagger F_\mu$, ensures that any of the three local outcomes (labeled by x, y or z) is useful,

$$F_z \equiv \sqrt{\frac{2}{3}}(|3/2_z\rangle\langle 3/2_z| + |-3/2_z\rangle\langle -3/2_z|) \quad (6)$$

$$F_x \equiv \sqrt{\frac{2}{3}}(|3/2_x\rangle\langle 3/2_x| + |-3/2_x\rangle\langle -3/2_x|) \quad (7)$$

$$F_y \equiv \sqrt{\frac{2}{3}}(|3/2_y\rangle\langle 3/2_y| + |-3/2_y\rangle\langle -3/2_y|), \quad (8)$$

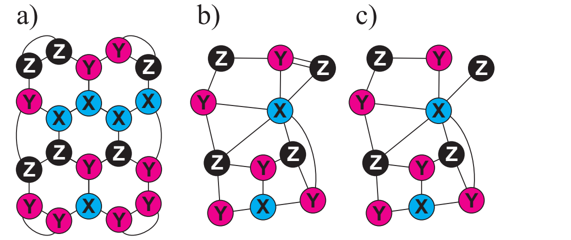

FIG. 2: Graphical rules for transformation of an AKLT state into an encoded graph state by the local POVMs. a) AKLT state on a honeycomb lattice $\mathcal{L}$, with a random pattern $\mathcal{A}$ of local POVM outcomes x, y, z. b) Edges of $\mathcal{L}$ with same-type endpoints are contracted. c) Edges of even multiplicity are deleted, edges of odd multiplicity are converted into standard edges. The resulting graph is $G(\mathcal{A})$.

where we have explicitly labeled the quantization axes. Thus, no matter what the outcome is, there is locally a two-level structure. The AKLT state has the rotational symmetry that is reflected in the selection of the three POVM elements. Once the measurement is carried out, the symmetry is broken randomly.

Whenever neighboring sites share the same outcome, labeled by μ, there are only two possible configurations for these two spins that can appear in the post-measurement state: $|3/2_\mu\rangle_A|-3/2_\mu\rangle_B$ and $|-3/2_\mu\rangle_A|3/2_\mu\rangle_B$. Physically, this reflects the antiferromagnetic properties of the AKLT state [7]. More importantly and generally, this means that when a set of connected sites share the same POVM outcome, they effectively constitute a single qubit. On the other hand, two neighboring sites that do not have the same outcome are two distinct qubits.

C. Encoded graph states

Using stabilizer formalism [13], we showed that for all POVM outcomes the resulting post-POVM state is a graph state. The detailed proof is in the technical version of this work [14]. The graph structure depends on the POVM outcomes, but can be constructed as follows; see Fig. 2 for illustration. First all connected sites that share the same POVM outcome will be collected in a set, which we call a *domain*. The domains are the vertices in the random graph, labeled by $G(\mathcal{A})$. If the number of the original honeycomb edges connecting two different domains is even, then in the random graph there is no edge between these two domains. However, if the number of honeycomb edges connecting two different domains is odd, then there *is* an edge between these two domains. This can be understood by the stabilizer formalism: intuitively, there is a $X - Z$ connection between two sites on the opposite domains.

Even though each domain may contain many physical spins, by local measurement, they can be reduced to a single one. The idea is simple. A GHZ-like state, $|00..0\rangle\otimes$

$|\psi_1\rangle + |11..1\rangle \otimes |\psi_2\rangle$, can be locally converted to one with one qubit fewer by measuring, say, the first qubit in $|0\rangle \pm |1\rangle$ basis. The post-measurement state, excluding the first qubit, becomes $|0..0\rangle \otimes |\psi_1\rangle \pm |1..1\rangle \otimes |\psi_2\rangle$. One can continues until $|0\rangle \otimes |\psi_1\rangle \pm |1\rangle \otimes |\psi_2\rangle$. Effectively, the domain can be regarded as a single physical qubit.

Our simulations showed that for a honeycomb lattice of $L \times L$ sites, the typical random graph contains macroscopic numbers of edges and vertices: (1) $0.496L^2$ vertices (2) $0.872L^2$ edges, respectively, as well as (3) $0.377L^2$ independent cycles. For details of our methods, see Ref. [14]. This indicates that there are sufficient number of qubit and entanglement left after the POVM.

D. Random graph states to 2D cluster states

By extending the proof in Ref. [15], we showed that if the two-dimensional random graph is in the connected phase (in the sense of percolation), the associated graph state can be locally converted to a 2D cluster state.

Intuitively, if the graph is in the connected phase, one can always identify a subgraph that has the topological structure of a 2D square lattice. One then uses local measurement in X, Y or Z basis to delete unwanted vertices and contract edges. The detail is referred to Ref. [14].

We performed Monte Carlo simulations and obtained that the typical random graphs are indeed deep in the percolated phase and in order to destroy the spanning structure of the graph one needs to delete randomly every edge with a probability at least $p_{\rm del}^{\rm (bond)} \approx 0.43$ or delete randomly every vertex with a probability at least $p_{\rm del}^{\rm (site)} \approx 0.33$. Therefore, the graph state can be converted to a 2D cluster state.

IV. CONCLUDING REMARKS

We have demonstrated that the 2D AKLT state, the ground state of an isotropic, two-body interacting Hamiltonian of spin-3/2, is a universal resource state for measurement-based quantum computation. Our approach applies to other 2D trivalent lattice, such as Archimedean lattices: $(3, 12^2)$, $(4, 6, 12)$ and $(4, 8^2)$, which have bond percolation thresholds greater than 2/3 [12]. In one dimension, our approach gives an alternative proof that the 1D AKLT state can be reduced to 1D cluster state, recently established by Chen et al. [16].

In showing that a state can be universal for MBQC, we have adopted the approach of establishing that the state can be converted to any of the existing known resource states [16–18] by local measurement, such as the cluster state [3]. We remark that there is an alternative route, i.e., by directly constructing a measurement scheme for universal gates, as was done in the original one-way computer [2] or the valence-bond approach [9, 19]. A recent beautiful work by Miyake [20] used this approach and utilized the fact that the bond percolation threshold on honeycomb lattice is below 2/3 to reach the same conclusion that the 2D AKLT state on the honeycomb lattice is a universal resource.

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