Under what conditions do quantum systems thermalize?

New insights from quantum information theory

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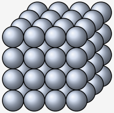
QIP 2011 - Singapore

How QIT helps understand many particle dynamics



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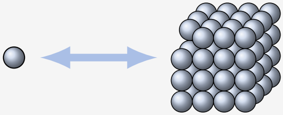

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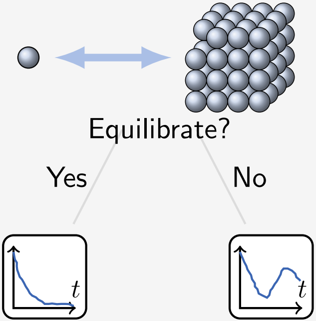

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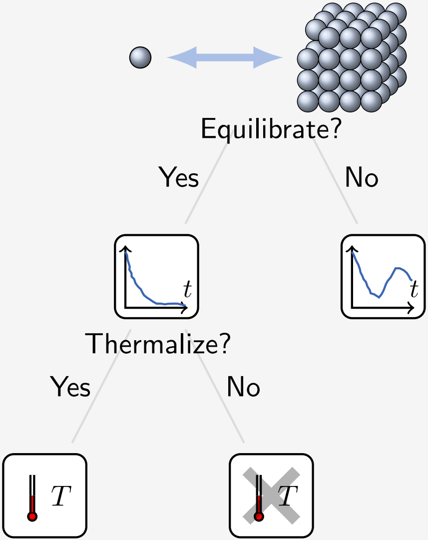

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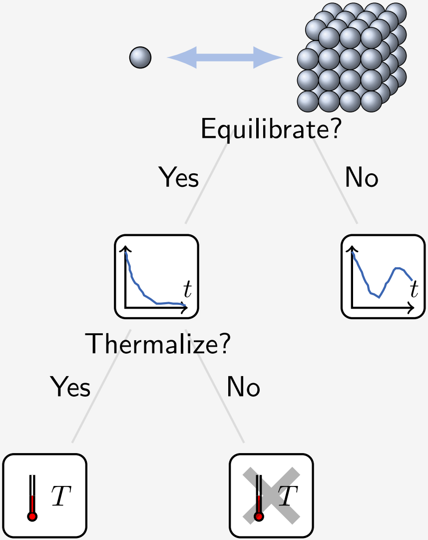

- Information propagation/ Lieb-Robinson bounds
- Haar-measure averages/ Concentration of measure

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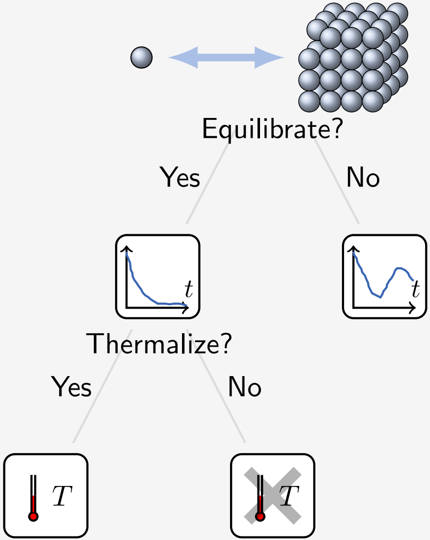

- Information propagation/ Lieb-Robinson bounds
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- Entanglement in the eigenbasis
- Perturbation theory

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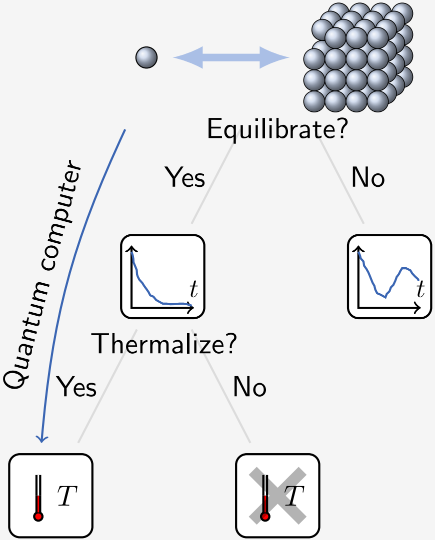

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Setup

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System, $\mathcal{H}_S, \mathscr{H}_S$

$d_S = \dim(\mathcal{H}_S)$

Bath, $\mathcal{H}_B, \mathscr{H}_B$

$d_B \gg d_S$

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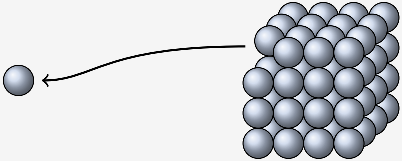

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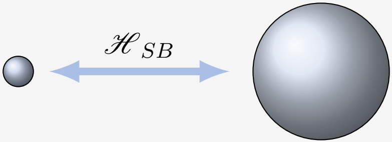

$$\mathscr{H} = \mathscr{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathscr{H}_B + \mathscr{H}_{SB}$$

$$\frac{d\psi_t}{dt} = \mathrm{i}\,[\psi_t, \mathscr{H}]$$

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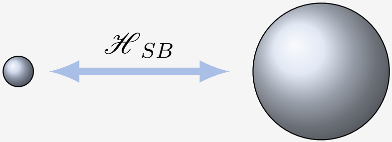

$$\psi_t^S = \mathrm{Tr}_B[\psi_t]$$

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$$\frac{d\psi_t}{dt} = \mathrm{i}\,[\psi_t, \mathscr{H}]$$

Equilibration

Equilibration

Theorem 1 (Equilibration [1])

If $\mathscr{H}$ *has non-degenerate energy gaps, then for every* $\psi_0 = |\psi_0\rangle\langle\psi_0|$ *there exists a* ω^S *such that:*

$$\overline{\mathcal{D}(\psi_t^S, \omega^S)} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\text{eff}}}}.$$

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103

Equilibration

Theorem

If $\mathscr{H}$ has

there exis

$\langle\psi_0|$

Non-degenerate energy gaps

$\mathscr{H}$ has non–degenerate energy gaps iff:

$$E_k - E_l = E_m - E_n$$

$$\Longrightarrow k = l \wedge m = n \quad \vee \quad k = m \wedge l = n$$

Intuition: Sufficient for $\mathscr{H}$ to be fully interactive

$$\mathscr{H} \neq \mathscr{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathscr{H}_2$$

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Equilibration

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Effective dimension

$$d^{\text{eff}} = \frac{1}{\sum_k |\langle E_k|\psi_0\rangle|^4}.$$

Intuition: Dimension of supporting energy subspace

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103

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$$\overline{\mathcal{D}(\psi_t^S, \omega^S)} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\mathrm{eff}}}}.$$

$\implies$ If $d^{\mathrm{eff}} \gg d_S^2$ then ψ_t^S equilibrates.

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103

Maximum entropy principle

Theorem 2 (Maximum entropy principle [2])

If $\mathrm{Tr}[A\,\psi_t]$ *equilibrates, it equilibrates towards its time average*

$$\overline{\mathrm{Tr}[A\,\psi_t]} = \mathrm{Tr}[A\,\overline{\psi_t}] = \mathrm{Tr}[A\,\omega],$$

and

$$\omega = \sum_k \pi_k \psi_0 \pi_k$$

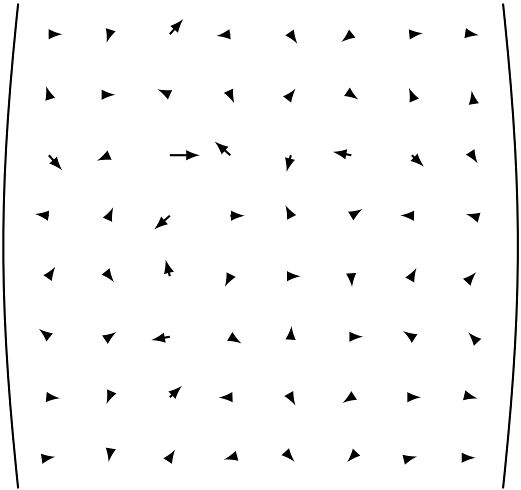
is the dephased state that maximizes the von Neumann entropy, given all conserved quantities.

[2] C. Gogolin, M. P. Mueller, and J. Eisert, (to appear in PRL), 1009.2493

Maximum entropy principle

Time averaging

$\psi_0 =$



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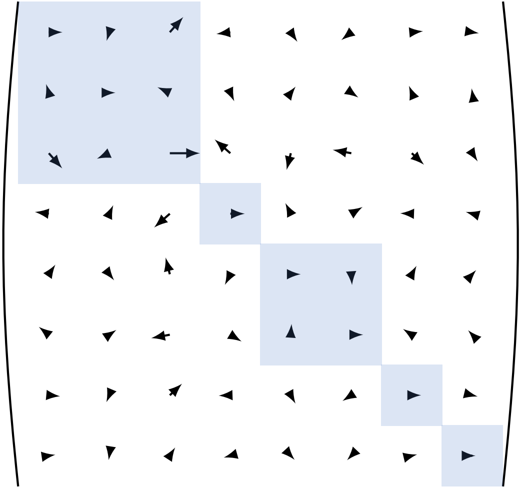

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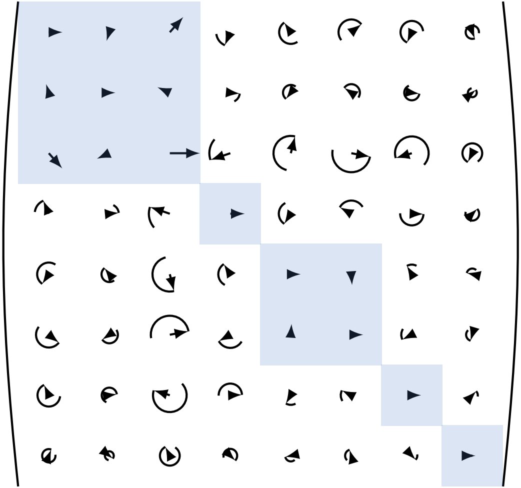

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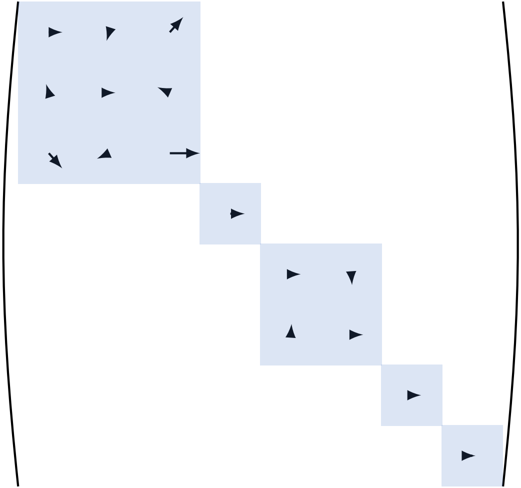

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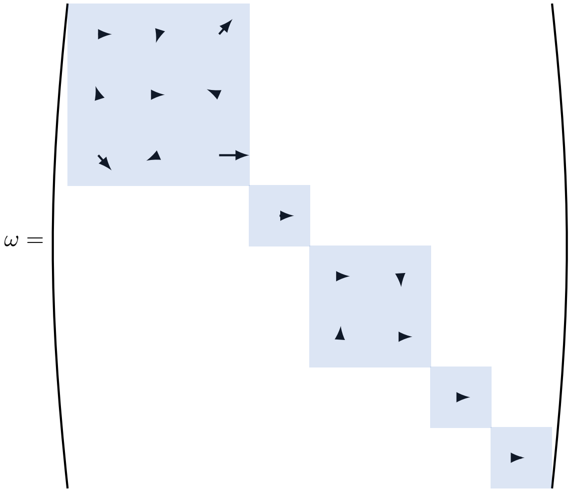

$\psi_0 \mapsto \omega$ is a pinching $\Rightarrow \omega$ maximizes entropy.

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is the dephased state that maximizes the von Neumann entropy, given all conserved quantities.

$\Rightarrow$ Maximum entropy principle from pure quantum dynamics.

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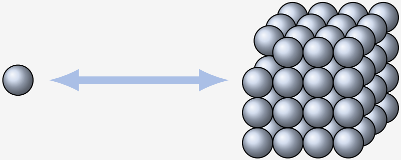
Thermalization

Thermalization is a complicated process



Thermalization implies:

1. Equilibration [1]
2. Subsystem initial state independence [2]
3. Weak bath state dependence [4]
4. Diagonal form of the subsystem equilibrium state [7]
5. Gibbs state $\omega^S = \mathrm{Tr}_B[\omega] \approx \mathrm{e}^{-\beta \mathscr{H}_S}$ [4]

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103
[2] C. Gogolin, M. P. Mueller, and J. Eisert, (to appear in PRL), 1009.2493
[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????
[7] C. Gogolin, PRE 81 (2010) no. 5, 051127

Level counting with no coupling

$$\mathscr{H}_0 = \mathscr{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathscr{H}_B$$

[3] S. Goldstein, PRL 96 (2006) no. 5, 050403

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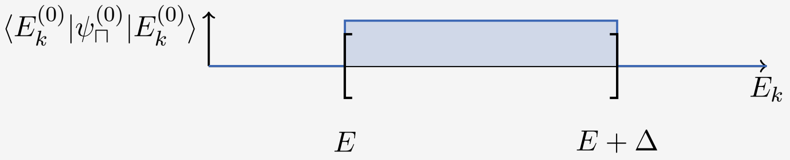

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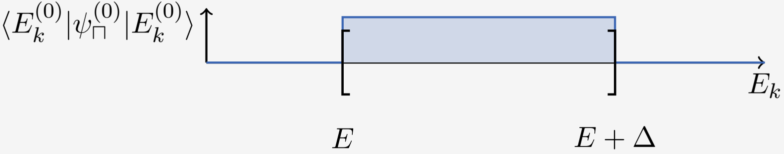

Well known fact [3]:

no coupling → # bath states in $[E - E_k^S, E - E_k^S + \Delta]$

$$\omega_\sqcap^{S(0)} \propto \sum_k \Omega_\Delta^B(E - E_k^S)\, |E_k^S\rangle\langle E_k^S|$$

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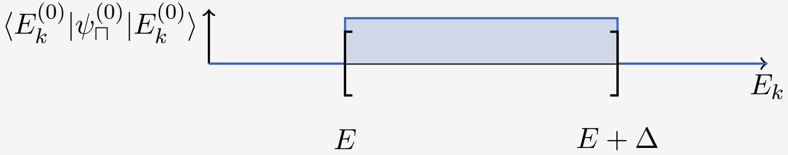

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exponentially dense spectrum

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[3] S. Goldstein, PRL 96 (2006) no. 5, 050403

Perturbative coupling $\| \mathscr{H}_{SB} \|_\infty < \text{gaps}(\mathscr{H}_0)$ ...

- ... is unrealistic as the spectrum of $\mathscr{H}_0$ becomes exponentially dense.

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Perturbative coupling $\| \mathscr{H}_{SB} \|_\infty < \text{gaps}(\mathscr{H}_0)$ ...

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- ... provably prevents thermalization because

perturbative coupling

$\Downarrow$ [2]

effective entanglement in the eigenbasis $R(\psi_0)$ is small

$\Downarrow$ [2]

absence of initial state independence.

$$\mathcal{D}(\omega^{S(1)}, \omega^{S(2)}) \leq \mathcal{D}(\psi_0^{S(1)}, \psi_0^{S(2)}) - R(\psi_0^{S(1)}) - R(\psi_0^{S(2)})$$

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$\Longrightarrow$ Refutes wide spread believe that "non-integrable models thermalize."

[2] C. Gogolin, M. P. Mueller, and J. Eisert, (to appear in PRL), 1009.2493

Realistic weak coupling

- Naive perturbation theory fails.
- Realistic weak coupling: $\mathrm{gaps}(\mathscr{H}_0) \ll \| \mathscr{H}_{SB} \|_\infty \ll \Delta$

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Theorem 3 (Corollary of a theorem from [4])

If $\| \mathscr{H}_{SB} \|_\infty \ll \Delta$ the dephased states $\omega_\sqcap^{S(0)}$ and $\omega_\sqcap^{S}$ are close to each other in the sense that

$$\mathcal{D}(\omega_\sqcap^{S}, \omega_\sqcap^{S(0)}) \lessapprox 3\sqrt{\frac{\| \mathscr{H}_{SB} \|_\infty}{2\Delta}}.$$

[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????

Consequences

$$\Longrightarrow \mathrm{Tr}_B[\omega_\sqcap] = \omega_\sqcap^S \approx \rho_{\mathrm{Gibbs}}^S$$

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Consequences

$$\Longrightarrow \mathrm{Tr}_B[\omega_\sqcap] = \omega_\sqcap^S \approx \rho_{\mathrm{Gibbs}}^S$$

"Theorem" 5 (Conclusion [4])

Assume $\Omega_\Delta^B(E - E_k^S)$ becomes exponentially dense to higher energies.

(Kinematic)
If the coupling is weak $\| \mathscr{H}_{SB} \|_\infty \ll \Delta$, almost all pure states from a microcanonical subspace $[E, E + \Delta]$ are locally close to a Gibbs state.

(Dynamic)
If the Hamiltonian has non-degenerate energy gaps all initial states $\psi_{\sqcap,0}$ with a flat energy distribution in $[E, E + \Delta]$ locally equilibrate towards a Gibbs state, even if they are initially far from equilibrium.

---

[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????

A quantum algorithm for Gibbs state preparation

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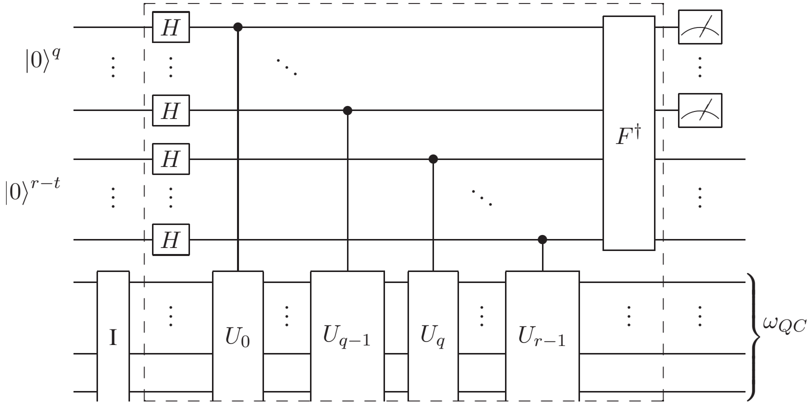
A quantum algorithm for Gibbs state preparation

Quantum circuit



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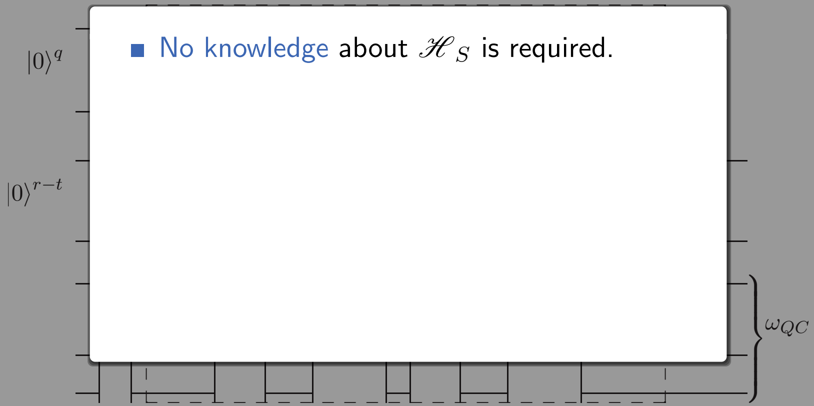

- No knowledge about $\mathscr{H}_S$ is required.

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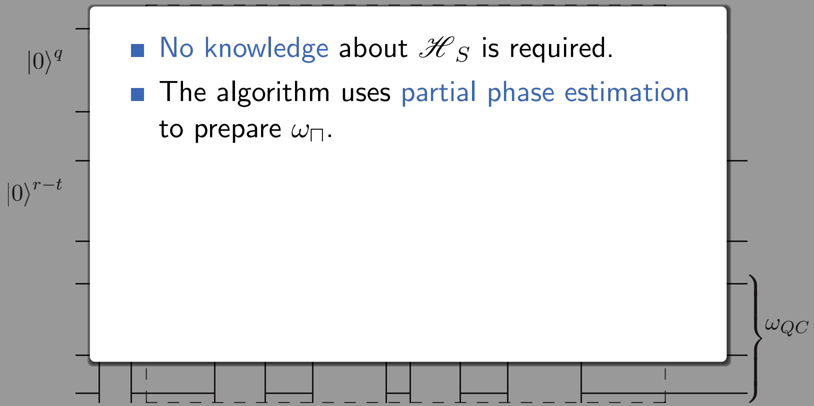

- No knowledge about $\mathscr{H}_S$ is required.
- The algorithm uses partial phase estimation to prepare $\omega_\sqcap$.

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A quantum algorithm for Gibbs state preparation

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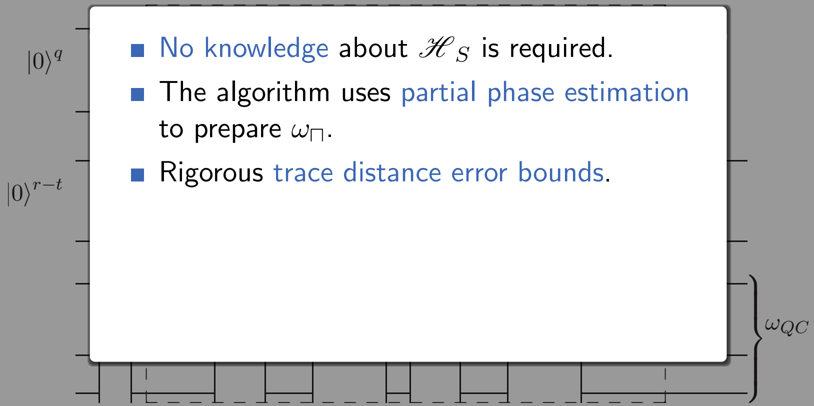

- No knowledge about $\mathscr{H}_S$ is required.
- The algorithm uses partial phase estimation to prepare $\omega_{\sqcap}$.
- Rigorous trace distance error bounds.

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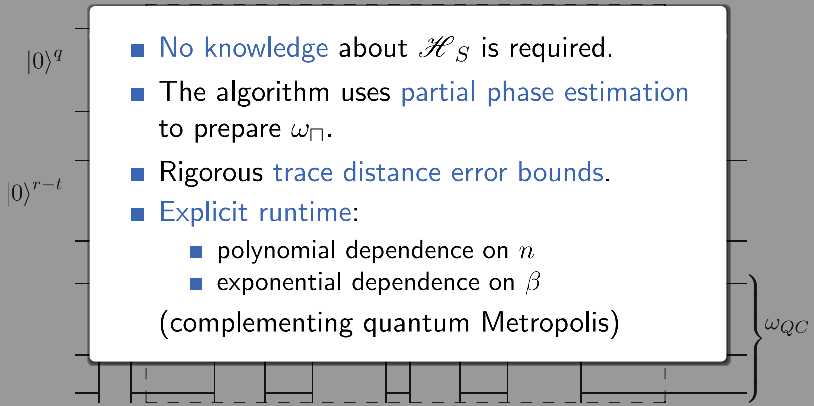
A quantum algorithm for Gibbs state preparation

Quantum circuit



- No knowledge about $\mathscr{H}_S$ is required.
- The algorithm uses partial phase estimation to prepare $\omega_{\sqcap}$.
- Rigorous trace distance error bounds.
- Explicit runtime:
  - polynomial dependence on n
  - exponential dependence on β

(complementing quantum Metropolis)

[4] A. Riera, C. Gogolin, and J. Eisert, (unpublished), 1101.????

Outlook

And there is more. . .

What I didn't talk about:

- Thermalization in exactly solvable models [5, 6]
- A strong connection to decoherence [7]
- Measure concentration [8, 1, 9, 10]

The major open question:

- Time scales. How long does it take to equilibrate/thermalize/decohere?

[1] N. Linden, S. Popescu, A. J. Short, and A. Winter, PRE 79 (2009) no. 6, 061103
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[6] M. Cramer and J. Eisert, NJP 12 (2010) 055020
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[8] S. Popescu, A. J. Short, and A. Winter, Nature Physics 2 (2006) no. 11, 754
[9] M. Mueller, D. Gross, and J. Eisert, 1003.4982
[10] C. Gogolin, Master's thesis, 2010, 1003.5058

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University of BRISTOL

CQT

Andreas Winter

References

Thank you for your attention!

$\longrightarrow$ slides: www.cgogolin.de

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