Entanglement can increase asymptotic rates of zero-error classical communication over classical channels

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The Q in the QIP for this talk:

$P^{E}_{\min}(M, \mathcal{E}^{\otimes n}) :=$ minimum $\Pr(\text{error})$ with entanglement assistance: Alice and Bob have shared entangled systems.

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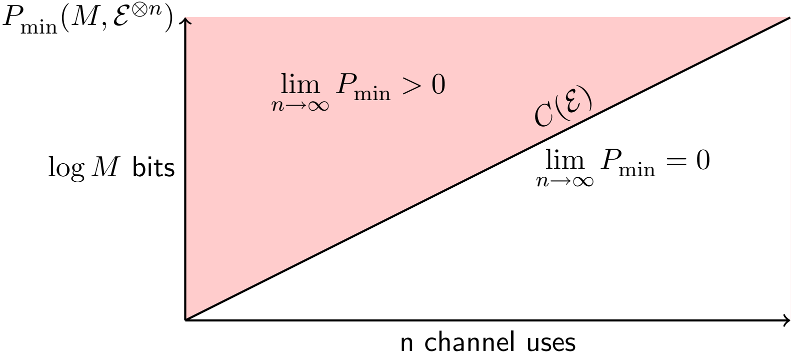
Capacity

$P_{\min}(M, \mathcal{E}^{\otimes n})$: Min. Pr(error) at rate $R = \frac{1}{n} \log M$ and block length n for classical codes.

Capacity



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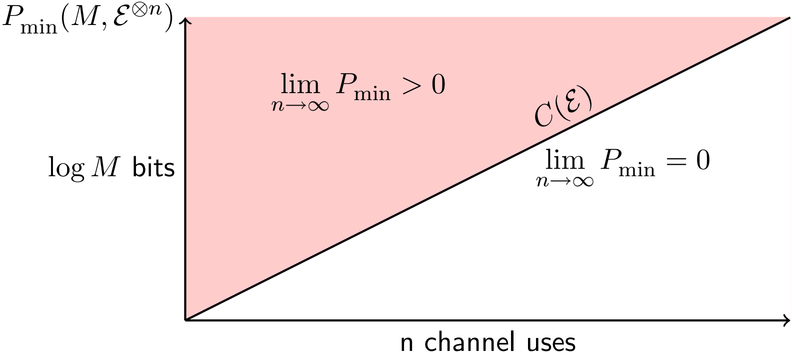

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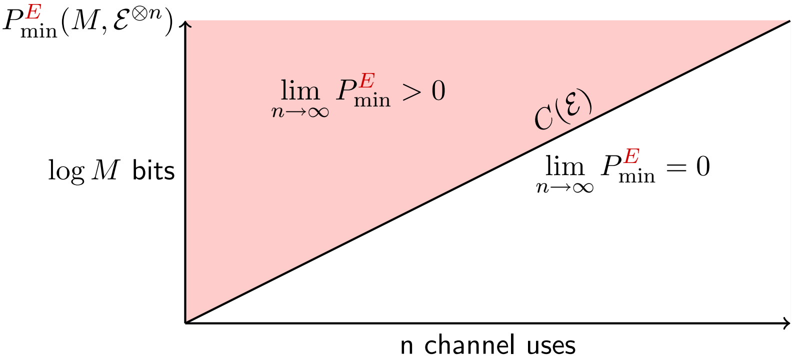

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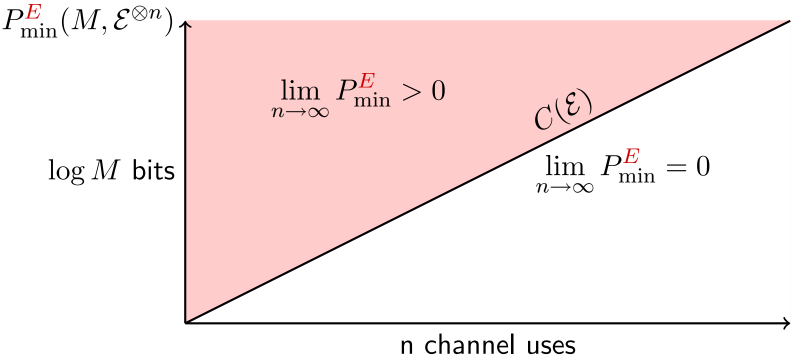

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- Not the whole story. . .

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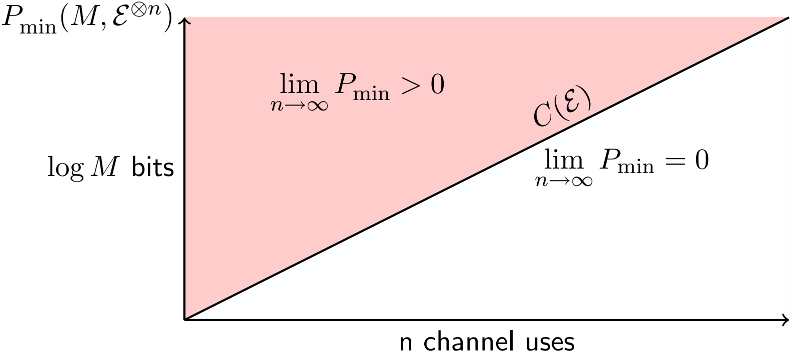

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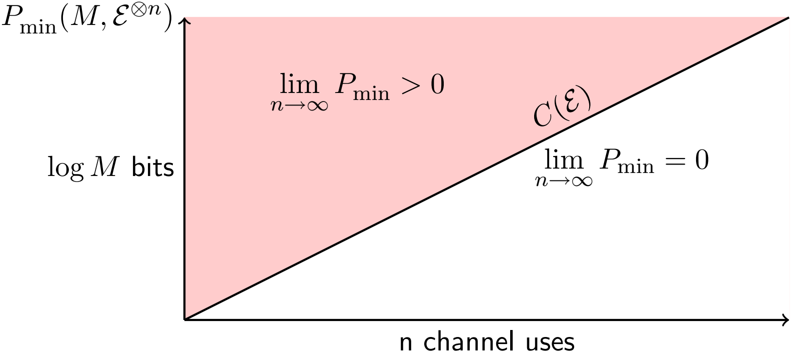

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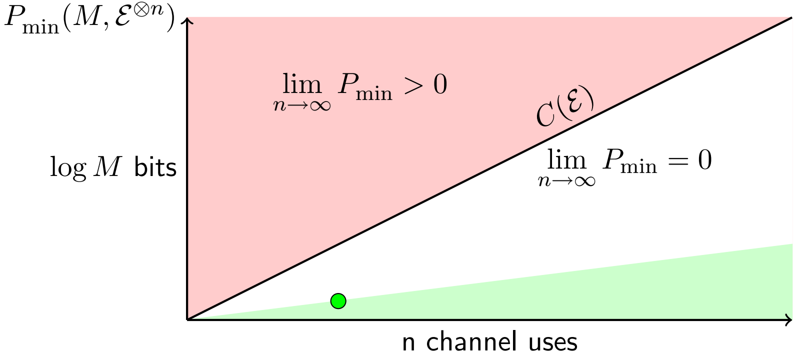

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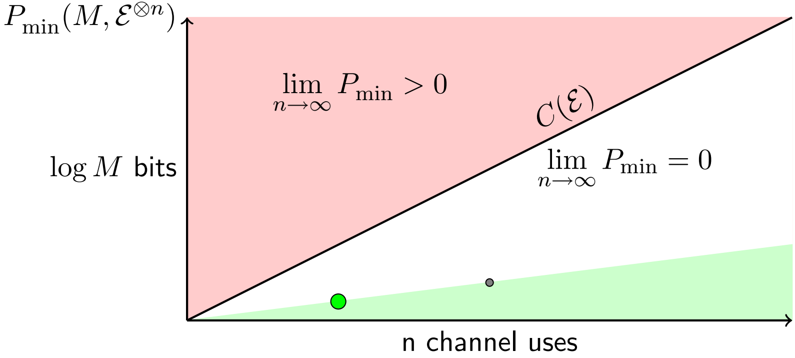

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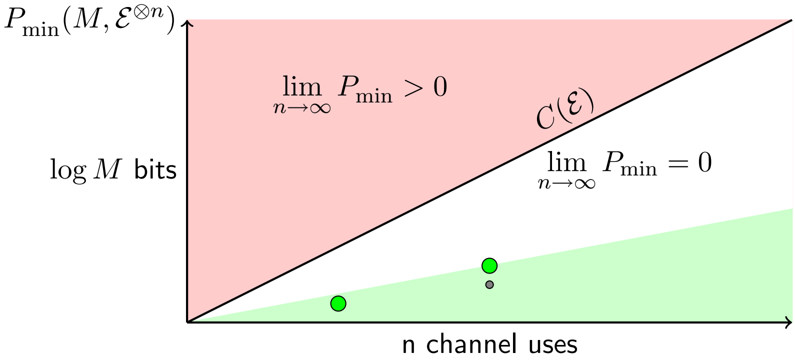

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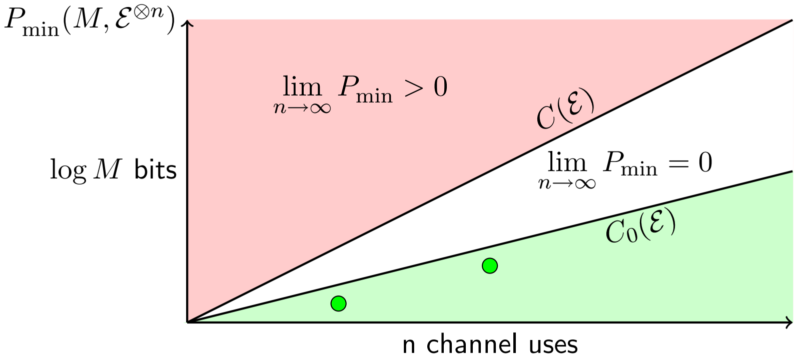

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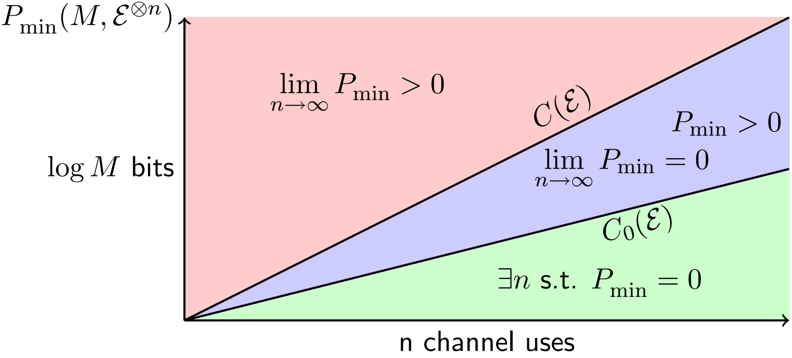

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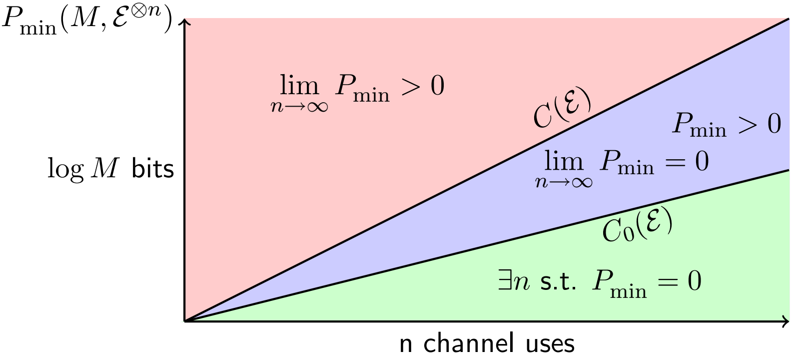

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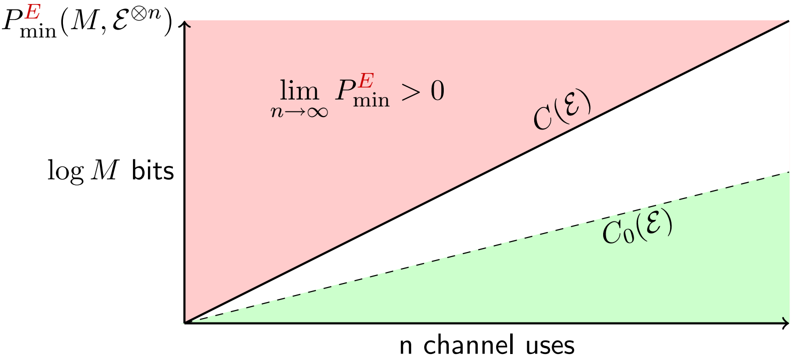
Entanglement-assisted zero-error capacity



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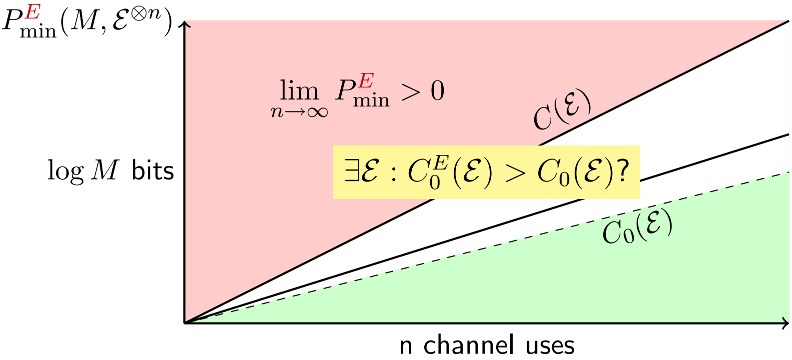

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Zero-error codes: Single channel use ($n = 1$)

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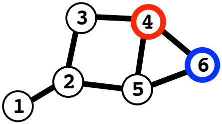

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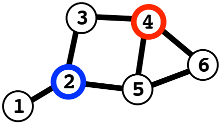

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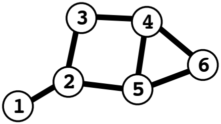

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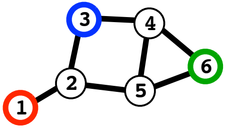

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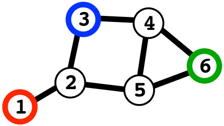

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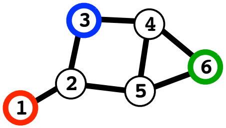

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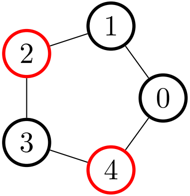
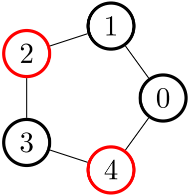


$M_0(C_5) = \alpha(C_5) = 2$

For $n = 1$, best zero-error rate is 1 bit/channel use.

Zero-error codes: Multiple channel uses ($n > 1$)

For composite channel $\mathcal{E}_1 \otimes \mathcal{E}_2$: Input sequences $(x_1', x_2') \neq (x_1, x_2)$ are adjacent iff x_i', x_i are adjacent or equal in $G(\mathcal{E}_i)$ for $i = 1, 2$.
$G(\mathcal{E}_1 \otimes \mathcal{E}_2) = G(\mathcal{E}_1) \boxtimes G(\mathcal{E}_2)$ (strong graph product).
$G(\mathcal{E}^{\otimes n}) = G(\mathcal{E})^n$: (strong graph power).



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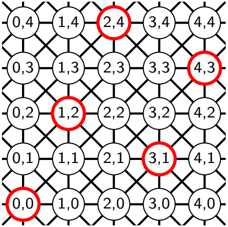

With two uses:
$M_0(C_5^{\otimes 2}) = \alpha(C_5 \boxtimes C_5) = 5$
For $n = 2$, best zero-error rate is $\frac{1}{2}\log 5 = 1.16\ldots$ bits/channel use.

Entanglement assisted zero-error protocols

Cubitt, Leung, Matthews, Winter 0911.5300 / 1003.3195

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If $G(\mathcal{E})$ has an orthonormal representation in $\mathbb{C}^d$ and partitions into m cliques of size d: $\kappa_1, \ldots, \kappa_m$. Then there is entanglement assisted protocol to send one of m messages with zero error. i.e. $M_0^E(G) \geq m$ and so $C_0^E(G) \geq \log m$.

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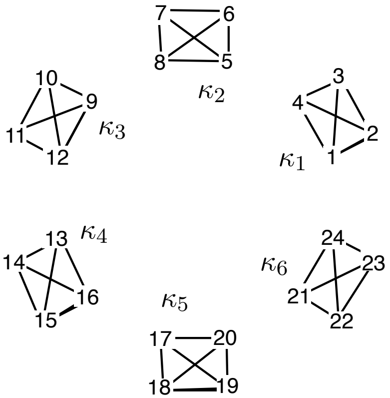
Goes like this (illustrated by $m = 6$, $d = 4$ case). . .

Entanglement assisted protocol for special graphs



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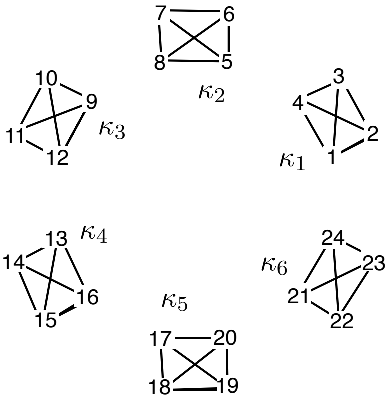

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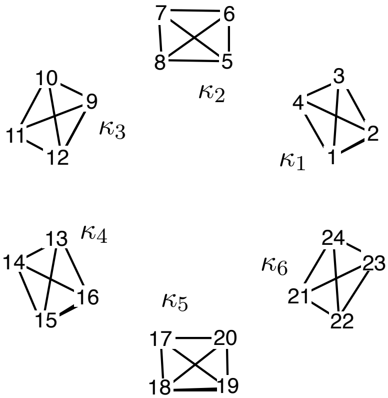

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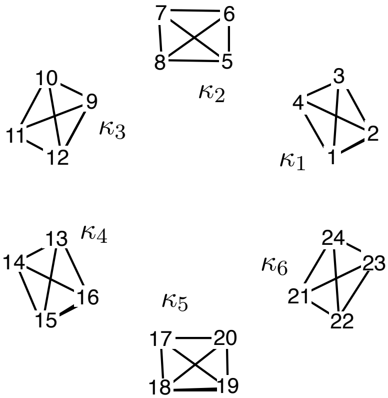

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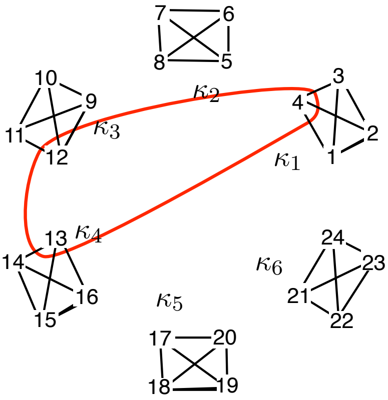

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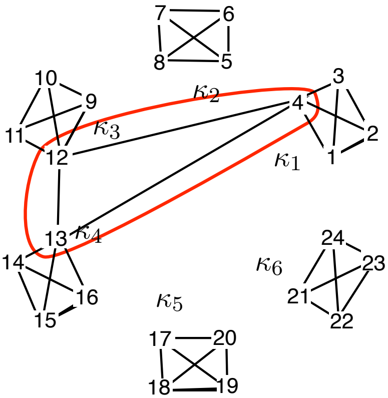

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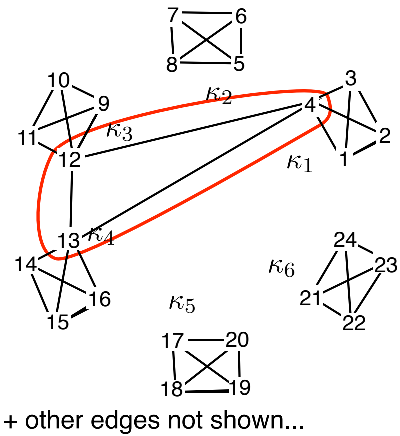

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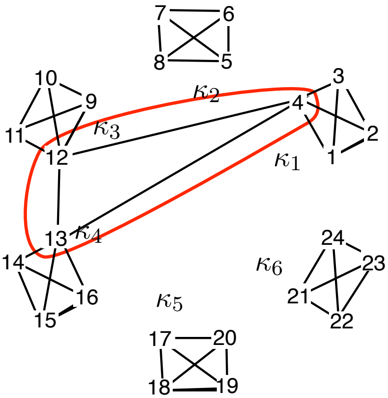

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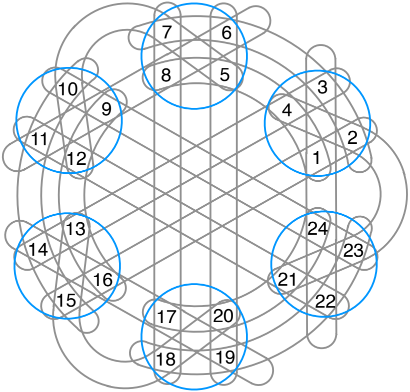
The graph F: Single use separation



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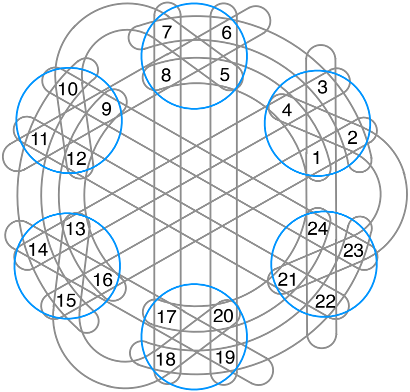

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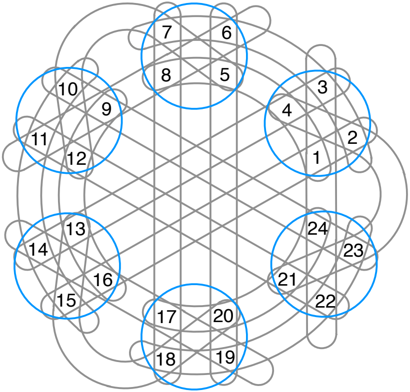

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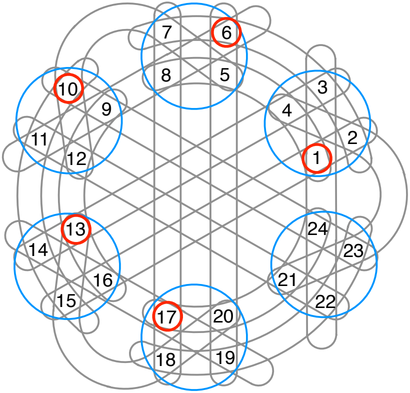

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C_0^E vs. C_0

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- Proof of separation will involve: Lower bounds on C_0^E and upper bounds on C_0.

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Theorem (ORCP Full: CLMW 1003.3195)

If G has orthonormal representation in $\mathbb{C}^d$ and partitions into m d-cliques then $C_0^E(G) = \log m$.

$C_0^E(F) = \log 6$. Nice, but only know $\log 5 \leq C_0(F) \leq \log 6$

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Matrix A (with rows and cols indexed by vertices) fits G if: $A_{xx} = 1$, $A_{xx'} = 0$ when x and x' not adjacent. E.g. for the pentagon:

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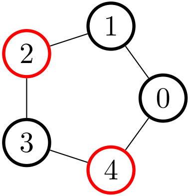

$$\begin{pmatrix} 1 & * & 0 & 0 & * \\ * & 1 & * & 0 & 0 \\ 0 & * & 1 & * & 0 \\ 0 & 0 & * & 1 & * \\ * & 0 & 0 & * & 1 \end{pmatrix}$$

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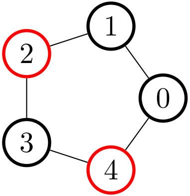

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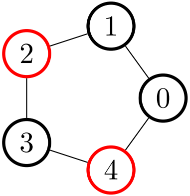

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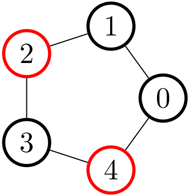

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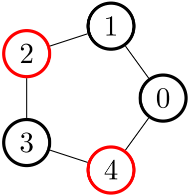

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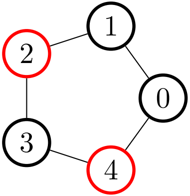

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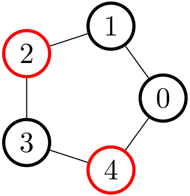

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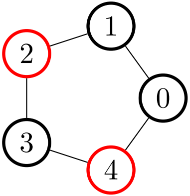

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- Idea: Look for graphs where Haemers bound $<$ Lovász bound

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If orthonormal rep. of Γ_j exists with dimension $2^j - 1$ then $C_0^E(\Gamma_j) = \log(2^j + 1)$ by Theorem ORCP. Can we find one for $j \geq 3$?

Root systems

We found one in an interesting place. . .

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Simple root systems: Finite set of vectors with certain special properties. 4 infinite families: A_n, B_n, C_n, D_n.

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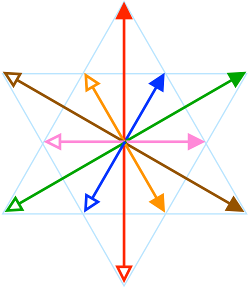

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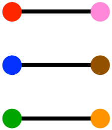

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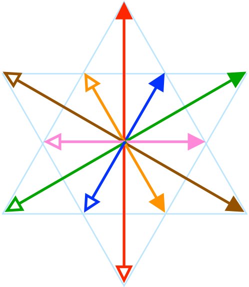

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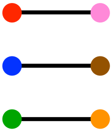

Orthogonality graph of d dimensional root system has d dimensional orthonormal representation.

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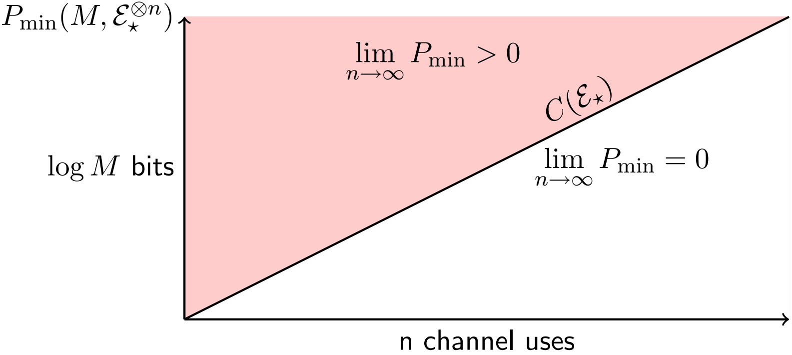
The channel $\mathcal{E}_\star$.



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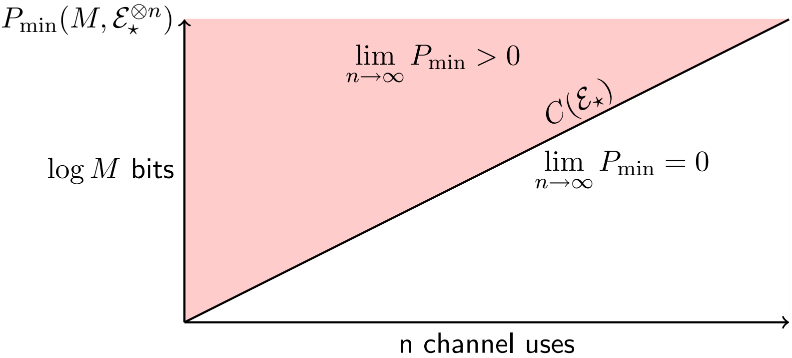

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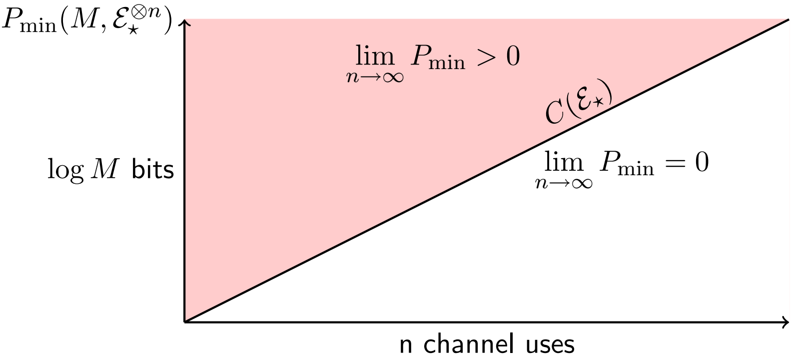

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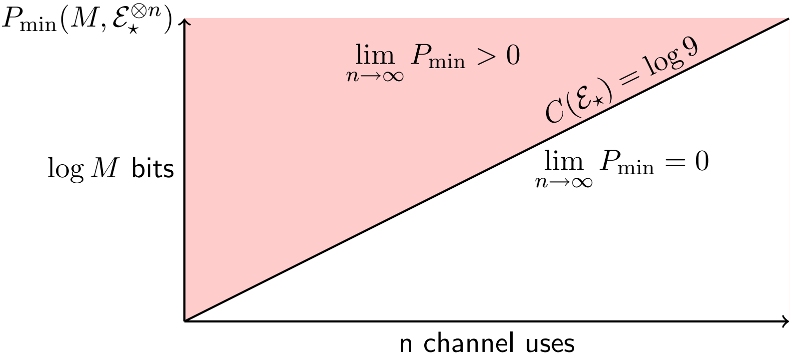

Capacity $C(\mathcal{E}_\star) = \log 9$.

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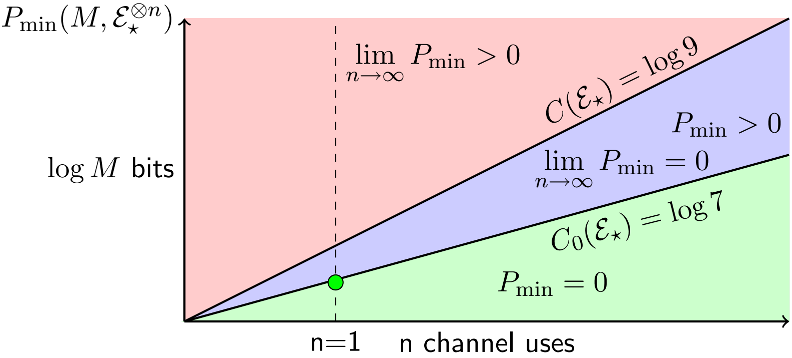

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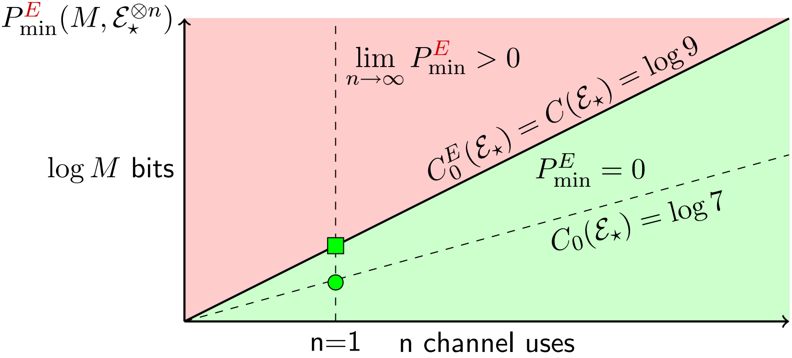

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Thanks!