The McEliece Cryptosystem Resists Quantum Fourier Sampling Attack

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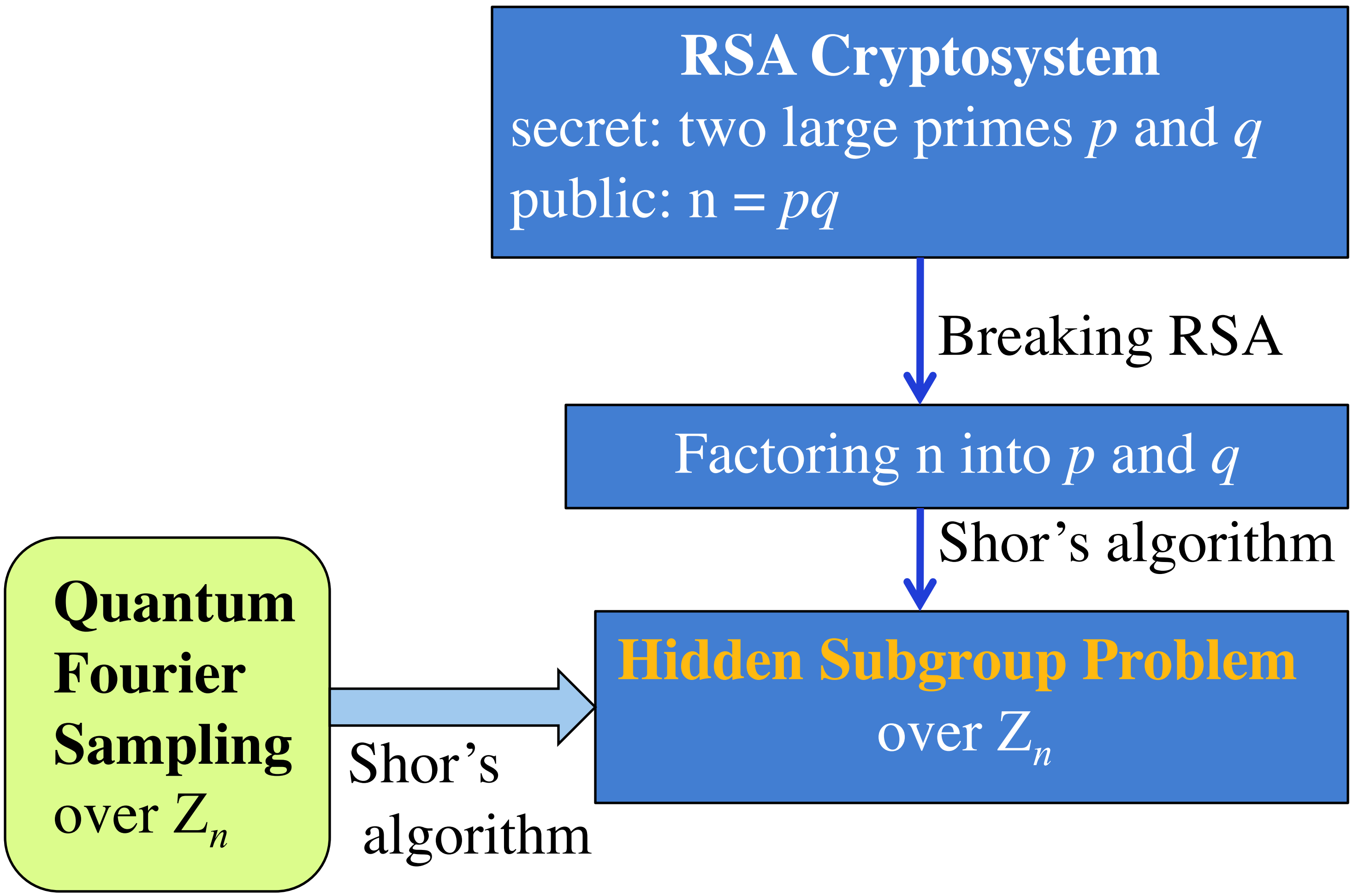
How RSA is Attacked by Quantum Computers



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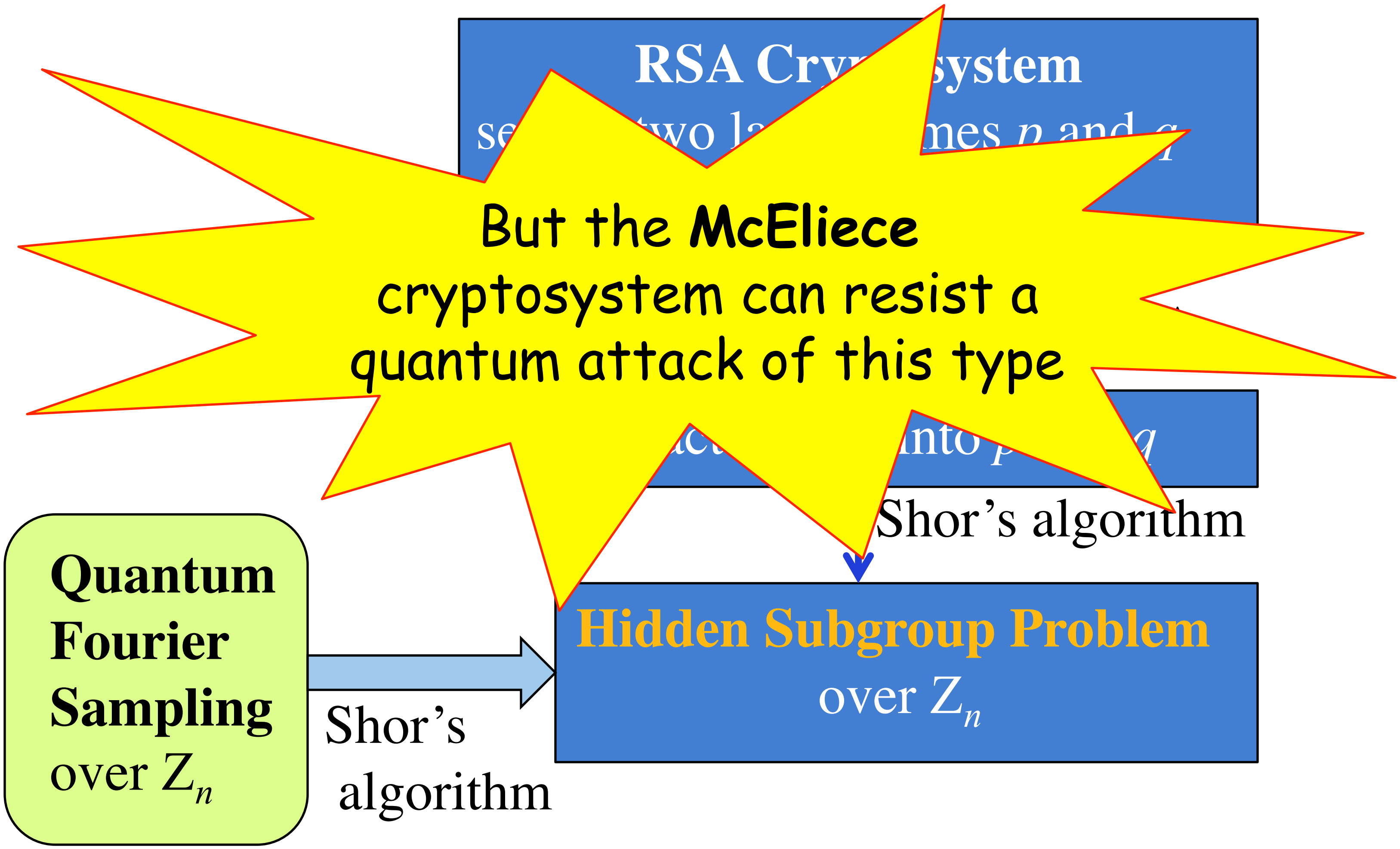


Hidden Subgroup Problem (HSP)

- HSP over a finite group G:
  - Input: function $f : G \rightarrow \{$■,■, …$\}$ that *distinguishes* the left cosets of an unknown subgroup $H < G$

  - Output: H
- Notable reductions to HSP:
  - Simon's problem reduces to HSP over $(Z_2)^n$
  - Shor's factorization reduces to HSP over Z_n
  - Graph Isomorphism reduces to HSP over S_n with $|H| \leq 2$

Quantum Fourier Sampling (QFS)

QFS over G to find hidden subgroup H:

Initial state

Use f

coset state

$$|gH\rangle = \frac{1}{\sqrt{|H|}} \sum_{h \in H} |gh\rangle, \text{ for a random } g \in G$$

Quantum Fourier transform

$$\sum_{\rho,i,j} \left(\rho(gH)\right)_{ij} |\rho, i, j\rangle$$

block matrix

Measure

weak → ρ

strong → ρ | i, j

The McEliece Cryptosystem

- Introduced in 1978 by Robert McEliece
- Based on error-correcting codes
  - decoding a general linear code is NP-hard.

- Long keys → require large storage
  - In 1978, not practical: 8KB RAM = $125 ☹
  - In 2011, no problem!: 2GB RAM = $30 ☺

- Considered secure classically
  - use binary Goppa codes, with good choice of parameters
  - leading candidate for post-quantum cryptography

The McEliece Cryptosystem
Key Generation

- Choose a secret linear code C
  - q-ary $[n,k]$-code that can correct t errors
- Private key:
  - M: $k \times n$ generator matrix of C
  - P: $n \times n$ random permutation matrix
  - S: $k \times k$ random invertible matrix over F_q
- Public key: (t, M*)

$$M^* = SMP$$

Scramble (S) — Permute (P)

A QFS Attack on McEliece Private Key

Given: M and M* = SMP → Recover: S and P

↓

Hidden Shift Problem over $GL_k(F_q) \times S_n$ with a hidden shift (S^{-1}, P)

↓

HSP over wreath product $(GL_k(F_q) \times S_n) \wr Z_2$ with a hidden subgroup H characterized by

- automorphism group $Aut(C)$ of the code C
- column rank r of M

nonabelian group

$$|H| \leq 2|Aut(C)|^2 q^{2k(k-r)}$$

How Strong is QFS?

- QFS over abelian groups
  - can be computed efficiently by quantum computers
  - That's how RSA is attacked!
- Recall:
  - the QFS attack on McEliece is over a nonabelian group
- Does QFS work over nonabelian groups?
  - Can QFS efficiently distinguish the conjugates of H from each other or from the trivial hidden subgroup?
  - No, in some cases.

Limitations of QFS over Symmetric group S_n

- Moore-Russell-Schulman, 2008
  - Strong QFS fails for any subgroup $H < S_n$ with $|H|=2$

- Kempe-Pyber–Shalev, 2007
  - Weak QFS fails for any subgroup $H < S_n$ unless H has constant minimal degree

the minimal number of points moved by a non-identity permutation in H

Our Results

- Strong QFS can't resolve the HSP reduced from the attack on McEliece private key if the secret code C is
  - well-permuted: $Aut(C)$ has large minimal degree and small order
  - well-scrambled: generator matrix M has large rank
  - Example:
    - rational Goppa code (generalized Reed-Solomon code)

Warning: This neither rules out other attacks nor violates a natural hardness assumption.

classically attacked by Sidelnokov-Shestakov: given M*=SMP, determine S and MP.

Our Results

- Strong QFS fails over S_n
  - even with hidden subgroups H of order > 2
    - extend Moore-Russell-Schulman's result
  - unless the minimal degree of H is $O(\log |H|)+O(\log n)$
    - prove a Kempe-Pyber–Shalev's version for strong QFS, though weaker in the upper bound on the minimal degree

- Strong QFS fails over $\mathrm{GL}_2(\mathrm{F}_q)$ if
  - H contains no non-identity scalar matrices, and $|H|=O(q)$
  - Example: H is generated by $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

Key Points of Our Proofs

- Generalize Moore-Russell-Schulman's framework
  - to upper-bound distinguishability of a subgroup $H<G$ by strong QFS over G.
  - Moore-Russell-Schulman's framework: $|H|=2$
  - Our framework: $|H| \geq 2$

difference between information extracted by strong QFS for a random conjugate of H and that for the trivial subgroup.

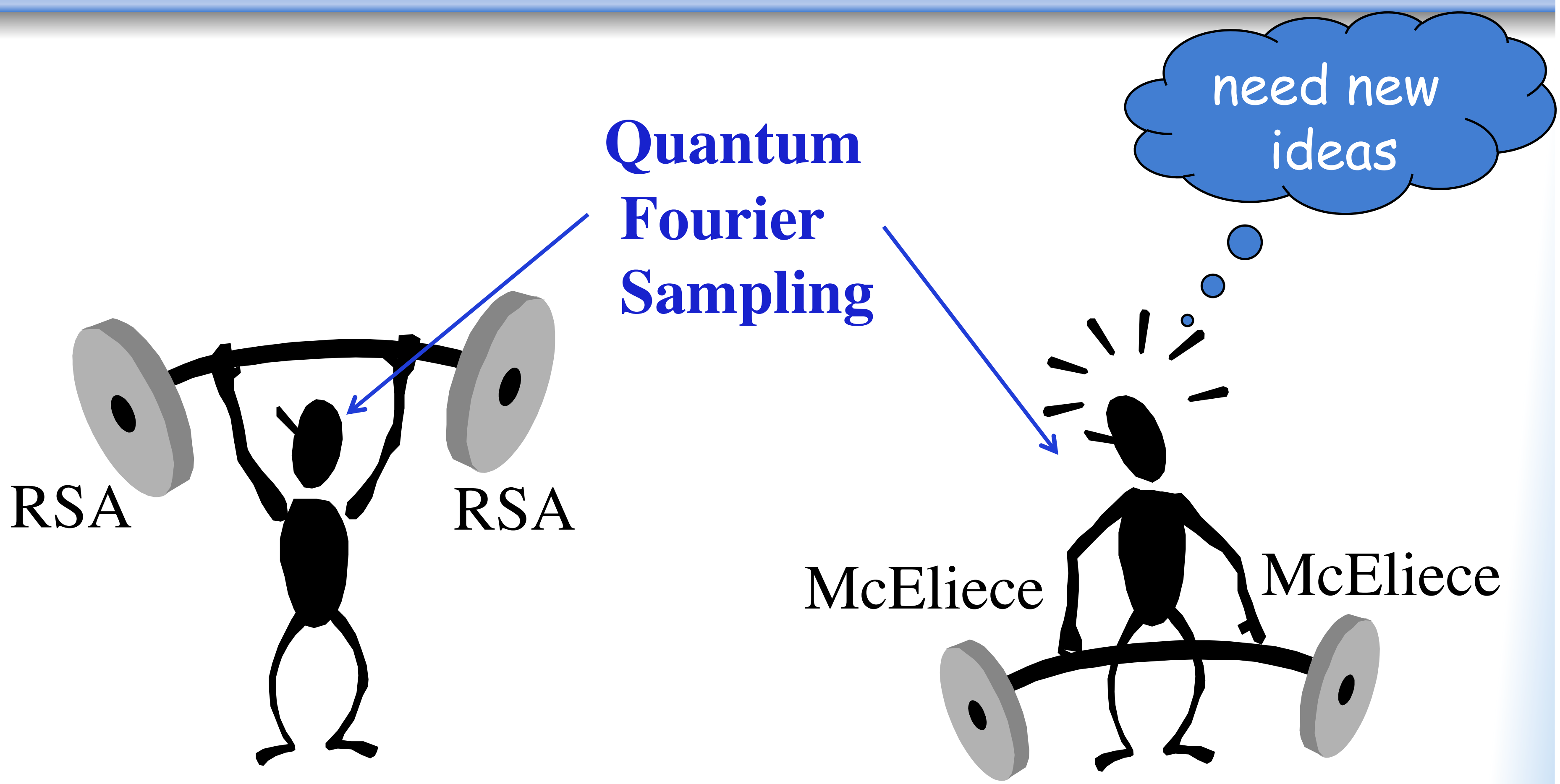
Key Points of Our Proofs

- Apply our general framework to
  - the HSP reduced from the McEliece cryptosystem
    → upper bound depending on
    - minimal degree of $Aut(C)$
    - order of $Aut(C)$
    - column rank of secret generator matrix M

Well-permuted, well-scrambled codes give good bounds

  - S_n and $\mathrm{GL}_2(\mathrm{F}_q)$

Conclusion



need new ideas
Quantum Fourier Sampling
RSA
RSA
McEliece
McEliece

Open Questions

- What are other linear codes that are well-permuted and well-scrambled?

- Can McEliece cryptosystem resist multiple-register QFS attacks?
  - Hallgren et al., 2006: subgroups of order 2 require highly-entangled measurements of many coset states.
  - Does this hold for subgroups of order > 2?

Questions?

Thank you!