Solution Space of Quantum 2-SAT

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SAT and Classical Complexity

- SAT: Satisfiability
  Can the values of a set of bits be assigned in such a way that they satisfy certain constraints?
- Complexity
  2SAT ------------- P
  3SAT ------------- NP-complete
  #2SAT ----------- #P-complete
  MAX-2SAT ------ NP-complete

QSAT

- QSAT: Quantum Satisfiability(Bravyi, 2006)

  Input: An integer n, a real number $\varepsilon = \Omega(1/n^{\alpha})$, and a family of k-qubit projectors $\{\Pi_S\}$, $S \subseteq \{1, . . . , n\}$, $|S| = k$.
  Promise: Either there exists n-qubit state $|\Psi>$ such that $\Pi_S |\Psi> = 0$ for all S, or $\Sigma_S<\Psi|\Pi_S|\Psi> \geq \varepsilon$ for all $|\Psi>$.
  Problem: Decide which one is the case.
- Can the quantum state of a set of qubits be chosen in such a way that it satisfies certain quantum constraints(that it is orthogonal to some other states)?

QSAT and Quantum Complexity

- Quantum Complexity

  Q2SAT --------------- P

  Q4SAT --------------- QMA_1-complete

  MAX-Q2SAT --------- QMA-complete

  Bravyi, 2006; Kitaev, Shen, Vyalyi, 2002; Kempe, Regev, 2003; Kempe, Kitaev, Regev, 2004; Nagaj, Mozes, 2006; Oliveira, Terhal, 2008.

  Random/generic SAT

  Bravyi, Moore, Russell, 2009; Laumann, et al, 2009; Ambainis, Kempe, Sattath, 2009; Laumann, et al, 2010; Movassgh, et al, 2010; Beaudrap, et al, 2010

Q2SAT

- 2SAT: P
- #2SAT: #P-complete
- Solution space structure: median graph

- Q2SAT: P
- Dimension of solution space of Q2SAT: ?
- Solution space structure: ?

Q2SAT in terms of Hamiltonian

- Let $H=\Sigma\, h_i$ be a two-local Hamiltonian on an n-qubit system. Each h_i is a projector (onto the constraints)
- Is the Hamiltonian frustration free: easy to answer as one ground state can be reduced to a product state (Bravyi,2006)
- What is the whole ground space like? How entangled can it be?
- Can it be reduced to a span of product states?
- What is the dimension of the ground space?

Our results

If the qubit 2-body Hamiltonian is frustration free

- There is always an ‘almost product’ ground state which contains at most two-qubit entanglement.
- The whole ground space can be reduced to a span of product states
- Counting the dimension of the ground space is #P-complete

Almost Product State

- How entangled can the least entangled ground state be in the whole ground space?
- How entangled can the unique FF ground state be?
- One-way quantum computer(Raussendorf,Briegel,2001)
- Unique FF ground state as universal recourse state
- Possible with spin-3/2(Cai, Miyake, Dur, Briegel, 2010; Wei, Affleck, Raussendorf, 2010; Miyake, 2010)
- Possible with spin-1/2?

Almost Product State

- Can there be unique FF ground state with multipartite entanglement?
- Two-qubit entanglement, yes

$$H = I - |\psi\rangle\langle\psi| \qquad |\psi\rangle = |01\rangle - |10\rangle$$

- No n-body entanglement for n>2
- Unique FF ground state can have at most 2-body entanglement.
- No-go for multipartite entanglement and resource state for one-way quantum computation

Almost Product State

- With degenerate ground space, the least entangled state has at most two-qubit entanglement.
- If a multipartite entangled state(n>2) is in the ground space, then there must also be a product state.
- Example: 3-qubit system

$$\left|GHZ\right\rangle = \left|000\right\rangle + \left|111\right\rangle \qquad \rho_2 = \left|00\right\rangle\left\langle 00\right| + \left|11\right\rangle\left\langle 11\right| \qquad h = \left|01\right\rangle\left\langle 01\right| + \left|10\right\rangle\left\langle 10\right|$$

$$h\left|000\right\rangle = 0, h\left|111\right\rangle = 0$$

$$\left|W\right\rangle = \left|001\right\rangle + \left|010\right\rangle + \left|100\right\rangle \quad \rho_2 = \left|00\right\rangle\left\langle 00\right| + (\left|01\right\rangle + \left|10\right\rangle)(\left\langle 01\right| + \left\langle 10\right|)$$

$$h = \left|11\right\rangle\left\langle 11\right| + (\left|01\right\rangle - \left|10\right\rangle)(\left\langle 01\right| - \left\langle 10\right|) \qquad h\left|000\right\rangle = 0$$

Almost Product State

- SLOCC equivalence: If $|\Psi>$ is the FF ground state of *H*, then V $|\Psi>$ is the FF ground state of $VHV^{\dagger}$.
  V is a tensor product of invertible operators on each qubit
- Suffice to consider one state for each equivalence class
- GHZ and W cover all 3 qubit states(Dur, Vidal, Cirac 2000)
- Induction to more qubits

Our results

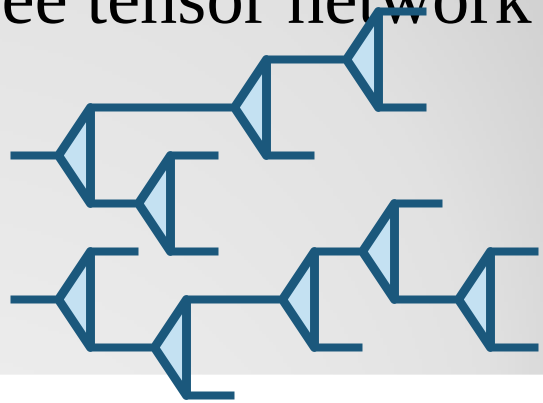
If the Hamiltonian is frustration free

- There is always an almost product state which contains at most two-qubit entanglement.
- The whole ground space can be reduced to a span of product states
- Counting the dimension of the ground space is #P-complete

Product Span

- Under reduction with isometries from two qubits to one qubit, one ground state can be reduced to product state. Bravyi, 2006
- Under such reduction, the whole ground space can be reduced to a span of product states
- Solution space is a span of tree tensor network states with the same tree structure.

  Perez-Garcia,Verstraete,Wolf,Cirac,2007;
  Shi,Duan,Vidal 2006;Vidal,2003;
  Beaudrap, Osborne, Eisert, 2010;



#Q2SAT is #P-complete

- Seems very classical
- Can we count ground space dimension as in the classical case?
- If the product states are supported on orthogonal local basis, e.g. 0 and 1, then Yes. #P-complete.
- True with some product constraint. Example

$$H = |\psi\rangle\langle\psi| \qquad\qquad |\psi\rangle = |00\rangle$$

Ground space spanned by |01>,|10>,|11>.

#Q2SAT is #P-complete

- Not with non-orthogonal product constraints

$$H = |00\rangle\langle 00|_{12} + |+0\rangle\langle +0|_{23}$$

- Solution: SLOCC into orthogonal basis

$$|0\rangle \rightarrow |0\rangle, |+\rangle \rightarrow |1\rangle$$

- Not with entangled constraint either.

$$H = |\psi\rangle\langle\psi| \qquad |\psi\rangle = |01\rangle - |10\rangle$$

Ground space spanned by |00>, |11>, |++>.

- Solution: map into product constraints with same ground space dimension.

#Q2SAT is #P-complete

- Counting the ground space dimension for Q2SAT can be converted into counting for 2SAT
- #Q2SAT is in #P
- #2SAT is #P-complete
- #Q2SAT is #P-complete

Conclusion and Outlook

- Our result
  - There always is an almost product solution
  - Solution space can be reduced to product span
  - #Q2SAT=#P-complete
- Not true beyond this point
  - Unique FF ground state with multipartite entanglement with two-body ***qutrit*** Hamiltonian (AKLT)
  - Also true with ***three-body*** qubit Hamiltonian (Cluster state on a chain)
  - With ***frustration***, everything becomes much harder(QMAc)